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A

Treatise on Arithmetic:

THROUGH WHICH

THE ENTIRE SCIENCE CAN BE MOST EXPEDITIOUSLY,
AND PERFECTLY LEARNED,

WITHOUT THE AID OF A TEACHER.

Designed for the Use of Schools and Private Students.

By NOBLE HEATH.

DEO GENERIQUE HUMANO.

PHILADELPHIA:
T. ELLWOOD CHAPMAN, No. 1 S. FIFTH ST.
1855.

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PREFACE.

HE who has, during the last forty years, been acquainted with the state of the science of Arithmetic, not only in the schools of this country, but in other parts, well knows that, at the commencement of that period, and for nearly half its duration, scarcely any thing better was found in the hands of the student than a miserable epitome or skeleton of the subject, presenting merely a few dogmas, followed by examples, without farther elucidation than the mechanical operation of one of these to serve as a model for the performance of all the others.

Divested almost entirely of scientific principles, Arithmetic was taught, in accordance with its title in the books then in use, as *the art of computing by numbers;* and a very clumsy, uninteresting, and wearisome art it was. Hence, the student, as might be expected, having no clue to the intricacies of the subject but memory, after ciphering through his Assistant,—as such a work was improperly called,—for the third, fourth, fifth, and even the sixth time, found himself almost as ignorant of the science as when he began.

Deeply felt and widely spread seem to have been the dissatisfaction, discouragement, and regret resulting from such palpable and enormous waste of the time and labour of the youth; for, during the last twenty years, new publications on this science, of much less exceptionable character, have, in the schools of Germany, France, England, and America, succeeded and superseded each other with surprising rapidity.

It is interesting to trace in these works the extremely gradual, yet steady approach toward a full development of the science, which, apparently, no one has hitherto dared to attempt.

The student who is early accustomed to investigation, in tracing effects to their cause, the dictamen of rules to the principles on which they are founded, the mutual dependence and connection of principles, and the skilful arrangement and combination of elements, in order to produce a desired result, thereby acquires a readiness and justness of perception, a conciseness in reasoning, a power of comprehension, and a strength of judgment unattainable by other means.

Experience having for a long time strongly impressed the mind of the author of this work with the above views, he was induced, many years ago, to impose on himself the arduous, and to him, from various causes, extremely protracted and difficult task of preparing, for schools and private students a copious yet compact treatise of Arithmetic, containing, together with an ample number of practical examples, a bold, connected, logical, and lucid developement of the principles and rules of the entire science. How far, in the persent work, that object is accomplished, the intelligent reader must determine.

SUMMARY OF REASONS FOR UNDERTAKING THIS WORK.

1. The almost total absence of every thing like theory in many arithmetics, and the very great imperfection and deficiency in this respect in the best of those now in use.

2. The utter impossibility that a master, however capable, who presides over a numerous class of pupils in various stages of advancement in the science, can find time to give to each a lucid verbal explanation to make up for the deficiency of his book.

3. The inefficiency of such verbal explanation, even if it could be given, for want of sufficient reiteration: said reiteration irksome in the extreme, and calculated to irritate the teacher to the discouragement of his pupil.

4. The inability of teachers to give the requisite explanation; for, without any disparagement to that self-devoted class of men, it is to be presumed that, taking the whole scope of the science, comparatively few are, impromptu, equal to the task.

BASIS OF THE WORK.

Whatever can be spoken can be written. Therefore, whatever could be said by the most enlightened teacher to his pupil, in explanation of every department of the science, the author has endeavoured to say in the book; and thus the scholar, having it before him, can read it as often as he pleases, to his own edification and great relief of the teacher.

SCHEME.

The subject is divided into six books, each of which exhibits a different department of the science. In each of the first four books, the cardinal operations are thoroughly discussed: in the first in Integers; in the second in Vulgar Fractions; in the third in Decimals; and in the fourth in Compound Numbers. The fifth book exhibits those rules which are not cardinal: viz., Involution, Evolution, Proportion, Progression, &c. &c., including Logarithms. The sixth book is devoted to mercantile science; in some departments of which, the author is greatly indebted to Kelly's Universal Cambist.

METHOD.

For the sake of easy reference, as well as logical concatenation, the work is subdivided into Sections and Articles, which are regularly numbered. Also, those passages which involve the most important principles, or leading points of the argument, particularly in the first four books, are italicized, the more perfectly to impress the mind of the pupil, and thus induce the habit of reflection.

Questions may be put by the teacher, to elicit from the pupil the passages italicized, or answers in accordance with them, thus:—TEACHER. What is quantity? SCHOLAR. *A limited portion of any natural object.*

This may be continued by the teacher throughout the work, which will be a good test of the progress of his pupil. The pupil, and he or she who is self-taught, may also pursue the same method for self-examination.

STYLE.

In presenting to the young, for daily contemplation, a subject of primary importance, much depends on the language in which that subject is invested.

A style elevated, rich, clear, and concise, being at once most appropriate and attractive, is best calculated to awaken and sustain a lively interest.

Prolixity wearies, puerility disgusts; both, therefore, as far as consistent with that most essential attribute, perspicuity, have been studiously avoided.

Brevity, wherever practicable, has been adopted, not only because it economizes time and expense, but because, by concentration, it lends force, and hence greatly facilitates comprehension.

ORIGINALITY.

The principle upon which this work is based, the plan upon which its greater divisions are arranged, and the method of conducting its details, both in calculation and language, combine to render it exceedingly unlike any of the popular works now in use.

Much that is entirely new in calculation will be found in the work, especially in the first, fourth, and fifth books. The rule for finding the least common multiple of several numbers, the author first gave in the year 1826, at which time he was employed as teacher in the Senior Department of the High School of New York.

SCOPE.

To the many novel features already presented, we may add the explanation of the nature and use of Logarithms, accompanied by a Table of those numbers for practical application. This, then, is probably the only school treatise which embodies the whole science, and hence the only one by means of which *the science can be perfectly learned without the aid of a teacher*.

Having thus endeavoured, instead of a barren, uninteresting, mechanical abstract, or an unwieldy, incongruous compilation, to give the subject the form, consistent with its important nature, of a transcendently beautiful science, it is hoped that the work here presented will amply reward the industrious student; and that, in its perusal, he will feel assured that he is, by a wholesome discipline, and consequent elevation of his mental power, preparing himself successfully to attempt the loftiest subjects to which human genius can aspire.

NOBLE HEATH.

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HEATH'S

TREATISE ON ARITHMETIC.

BOOK I.

PRELIMINARY IDEAS—DEFINITIONS AND FORMATION OF NUMBERS—NUMERATION, ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF WHOLE NUMBERS.

SECTION I.

Preliminary Ideas, Definitions, and Formation of Numbers.

1. *A limited portion of any natural object;* as of time, space, weight, heat, &c., or of any substance, fluid, solid, or æriform, is called *quantity.*

2. The object of all mathematical science is *to measure quantity.*

3. Quantity can only be measured *by comparing it with some known quantity of the same kind acknowledged as a standard.*

4. Standard Measures, to prevent error or variableness, are generally derived *from nature.* For example, measures of time, *from the time of the revolution of the earth about its axis:* of space, *from the length of a barley-corn,* taken from the middle of a full-grown ear; also, *from the circumference of the earth:* of weight, *from the weight of a grain of wheat,* taken as above; also, *from the weight of a definite quantity of distilled water:* of heat, *from the temperature of boiling water,* &c.

5. Quantity, mathematically considered, is only susceptible *of increase and diminution;* these are, therefore, called *the attributes of quantity.*

6. A quantity may be increased in two ways: either *by*

combining with it a quantity of the same kind, greater or less than itself, which is called *Addition;* or, *by combining with it, or continuing to add to it, a quantity exactly equal to itself,* called *Multiplication:* because, in the latter case, the quantity is repeated many times.

7. A quantity may also be diminished in two ways, either *by taking from it a quantity not greater than itself,* called *Subtraction;* or, *by repeatedly taking from it a quantity less than itself,* called *Division:* because, by this last operation, the quantity is evidently separated or divided into a number of equal parts.

8. The two attributes of quantity, viz., increase and diminution, are then the two *fundamental principles,* upon which are based *the four operations—Addition, Subtraction, Multiplication, and Division.*

9. Now, as there is no other method whereby a quantity can be increased and diminished, these four operations are called *the four cardinal or fundamental operations of Arithmetic:* all others being dependent on and performed by them.

10. Arithmetic, *the science of numbers,* then, is, in its most extended sense, neither more nor less than *the knowledge of all the numerical operations based upon, and arising out of, the two simple attributes of quantity—increase and diminution.*

11. The four cardinal operations are represented by the following signs:

+ *Plus* or more, the sign of Addition.
— *Minus* or less, " Subtraction.
× or • *Into,* " Multiplication.
÷ *By* (divided by) " Division.

12. That which we attempt to examine, discuss, determine, or prove, by one or more mathematical operations, is called *the Proposition;* commonly in Arithmetic, *the Question.*

13. The truth or conclusion arrived at is called *the Result* or *Answer.*

14. The general process by which a proposition or question is determined is called *the Solution.*

15. When, in solving a question, only one cardinal operation is used, *the answer has a distinctive name.*

In Addition, the answer is called the *Sum.*
Subtraction, " " *Difference* or *Remainder.*
Multiplication, " " *Product.*
Division, " " *Quotient.*

Thus: a plus b $(a + b)$ is *a Sum*, (a added to b.)
a minus b $(a - b)$ is *a Difference*, (b taken from a.)
a into b $(a \times b)$ is *a Product*, (a multiplied by b.)
a by b $(a \div b)$ or $\frac{a}{b}$ is *a Quotient* (a divided by b.)

16. The quantity established as a Standard Measure is called *a Unit*. Thus, *One Barley-corn* is *a Unit.* Three barley-corns, placed lengthwise in contact, make *one inch;* twelve inches, *one foot;* three feet, *one yard,* &c. *The time of the revolution of the earth about its axis* is called *one day.* The day is divided into twenty-four equal parts, each of which is *one hour.* The hour is divided into sixty equal parts, each of which is *one minute.* The minute is divided into sixty equal parts, each of which is *one second.*

17. The Unit is represented *by the Figure* 1, one; which, in its general sense, signifies *one thing of any kind;* and serves as a *Standard of Comparison* for all things of the same kind. Hence the Figure 1 may represent 1 barley-corn, 1 inch, 1 foot, 1 yard, 1 day, 1 hour, or any other known standard measure or thing.

18. The figure 1 is *still a unit,* when not applied to any thing, in which sense it is called *Abstract,* (*abstrahĕre,* Lat., *to separate.*) When applied, it is called *Concrete,* (*concretus,* Lat., *joined together.*)

19. *The Combination or Addition of two or more Units* forms what is called *a Number.*

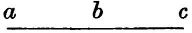
20. A sign made thus $=$, called *Equal to* or *Equals,* is placed between two quantities or expressions to show their equality; the whole forming what is called *an Equation.* Thus, the Equation $1 + 1 = 2$ is read, *one plus one, equal to two;* or, more commonly and perhaps better, *one plus one, equals two.* Hence, (Art. 19,) 2 is the first or lowest number.

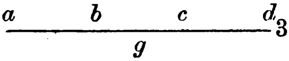
A sign made thus $>$ is called *greater than;* reversed thus $<$, *less than.*

Observe that $1 + 1$ is *a Combination* or *Addition,* in fact *a Sum* or *Product;* and this *sum* or *product* equals 2. It is erroneous to say $1 + 1$, one plus one, *are* equal to 2; or 1 and 1 *are* equal to 2, because the plural word (*are*) implies their disunion. We must therefore say 1 and 1 *is* 2, or twice 1 *is* 2. That is, *the sum* or *product* is 2.

21. The next Combination, formed by adding another unit, is expressed by each of the Equations $1 + 1 + 1 = 3$;

$2+1=3$, or $1+2=3$; which we read 1 plus 1 plus 1 equals *three;* 2 plus 1 equals 3, or 1 plus 2 equals 3.

22. Let the line 

be 2 inches long, and the line 

3 inches long: then, the middle of *a b c* being at *b*, and the middle of *a b c d* at *g*, half-way between *b* and *c*, it is plain that we can divide *a b c* evenly; that is, into 2 equal parts, without breaking either of its units. Also, that we cannot so divide *a b c d;* because the unit *b c* will be broken at *g;* and each half of the line *a b c d* will contain *one inch and a half.*

23. A number consisting of whole units only is called *an Integral* or *Whole Number,* (*integer,* Lat., *whole.*) A number containing broken parts is called *a Broken* or *Fractional Number,* (*fractus,* Lat., broken.) Hence the numbers 2 and 3 are *whole numbers;* but one and a half is *a fractional number.*

24. A number whose half is integral is called *an even number.* And a number whose half is fractional is called *an Odd Number.* Hence 2 is *an even number;* and 3, *an odd number.*

25. Beginning with the Unit, (which, though in itself, really not a number, is often, for the sake of form, called *Number* 1,) we progress regularly *by adding a unit at a time,* in forming what is called, *the Scale of Natural Numbers:* thus, 1 *one,* 2 *two,* 3 *three,* 4 *four,* 5 *five,* 6 *six,* 7 *seven,* 8 *eight,* 9 *nine,* &c. It is easy to see that, in this scale, *the numbers are odd and even ad infinitum* (*ad infinitum* or *ad inf.,* Lat., *to infinity* or *without end.*)

26. Though the Scale of Natural Numbers evidently progresses ad inf., seeing that, *however great a number may be, we can still add a unit,* the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, are the only ones by which any number, whole or fractional, however great, is expressed; from which it is easy to infer that *each of these figures is made to assume new values, increasing ad inf.*

27. The next number in the scale, called *ten,* is expressed by the figure 1, which assumes this new value *merely by a new position or place,* and to show this the figure 0, *Nought* or *Cipher,* (which signifies *nothing,*) is placed on the right of it, thus: 10. Because 0 has no value or signification, except *merely empty place,* the other figures are, by way of distinction, called *Significant Figures;* sometimes *Digits,*

(*digitus*, Lat., *finger*.) The first idea of the method of expressing numbers having been, it is supposed, derived *from the fingers*.

28. In the number 10, the nought is considered as occupying *the place which the digits occupy when alone;* and, therefore, this first, or right-hand place, is called *the place of units.* The unit being thus removed one place towards the left, has, *merely from its place*, assumed a new value equal to *ten times its former value;* and, hence, the place where it stands, or second place, is called *the place of tens.*

29. The place of units being empty, we continue the regular formation of numbers, by substituting, for the nought, the nine digits in succession: thus, beginning with ten, we have the numbers 10 *ten*, 11 *eleven*, 12 *twelve*, 13 *thirteen*, 14 *fourteen*, 15 *fifteen*, 16 *sixteen*, 17 *seventeen*, 18 *eighteen*, 19 *nineteen:* the numbers 11, 12, &c. being *ten* and *one*, *ten* and *two*, *ten* and *three*, &c. to *ten* and *nine*, inclusive.

30. Continuing the formation, we add 1 to the 9 units of 19; and transferring the *ten*, thus formed, to the second place, *by a unit*, as before: we add this 1 to the 1 already there, which makes 2 tens, or 20, *twenty*, and the place of units again becomes vacant. Then, repeating the digits in this place, as before, we form the numbers 21 *twenty-one*, 22 *twenty-two*, &c. to 29 *twenty-nine.*

31. By adding 1 to the 9 units of 29, and transferring the ten, thus formed, to the second place, we have the number 3 *tens* or 30 *thirty*. Continuing thus, we form the tens in succession, which are 10 *ten;* 20 *twenty;* 30 *thirty;* 40 *forty;* 50 *fifty;* 60 *sixty;* 70 *seventy;* 80 *eighty;* 90 *ninety*. Again repeating the digits, we have 91 *ninety-one*, 92 *ninety-two*, &c. to 99 *ninety-nine*, which is *the greatest number expressed by two figures.*

32. By adding 1 to the 9 units of 99, and carrying the ten to the second place, by the figure 1, as usual; this, together with the 9 already in this place, makes ten tens, which we express *precisely in the same manner as we did ten units*, namely, by the figure 1 carried to the next place towards the left, by which means the place of tens, as well as the place of units, becomes vacant, and we supply both accordingly with ciphers, thus: 100, and read *one hundred*. The third place is, therefore, called *the place of hundreds.*

33. Thus we see that the use of the cipher is *to hold the place of an order of units which becomes vacant.*

34. Continuing the formation of numbers in the empty places, as at first, we have 101 *one hundred and one;* 102 *one hundred and two,* &c., to 199 *one hundred and ninety-nine.* Again, proceeding as in Art. 32, we form the number 200 *two hundred;* and in like manner, as above, successively, the numbers 300 *three hundred;* 400 *four hundred;* 500 *five hundred;* 600 *six hundred;* 700 *seven hundred;* 800 *eight hundred;* 900 *nine hundred;* as also the numbers 901 *nine hundred and one;* 902 *nine hundred and two,* &c., to 999 *nine hundred and ninety-nine;* which is *the greatest number expressed by three figures.*

35. In adding a unit to 999, to form the next number, we have first *ten units,* which we express *by* 1 *in the second place;* then *ten tens,* which we express *by* 1 *in the third place;* and, lastly, *ten hundreds,* which we express *by* 1 *in the fourth place;* thus, 1000, (the three vacant places being supplied with ciphers,) and read *one thousand.* It is easy to see that, proceeding as above, we should successively form the numbers, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000; which are read *two thousand, three thousand,* &c. to *nine thousand;* as well as the numbers 9001 *nine thousand and one;* 9002 *nine thousand and two,* &c., to 9999 *nine thousand nine hundred and ninety-nine.*

36. Hence we see that *the formation of new orders towards the left is without end,* depending upon this universal law, namely—*the value of ten units of any order is expressed by one unit of the next order towards the left*—according to which, the digits assume new values, increasing towards the left ad inf. *tenfold for each remove,* and are, by this means, capable of expressing any number whatever.

37. We have seen that the value of a unit in the second place is equal to $9+1$; in the third, to $99+1$; in the fourth, to $999+1$, &c., according to a general law: wherefore, the value of a unit, in any place towards the left, is equal to *the number expressed by a horizontal row of nines equal, in number, to the number of places on the right of that unit, plus* 1: and, hence, a unit in any place towards the left expresses *a number greater than that expressed by all the figures which can stand on the right of it.*

SECTION II.

NUMERATION.

38. Numeration is *the general method of reading and writing numbers.*

39. As there is no end to the formation of new orders on the left, and as it would, even in ordinary numbers, be inconvenient to give a new name to each new order, the figures of large numbers are separated *into periods of three figures each.* Thus, beginning *at the right hand*, and proceeding towards the left, *we separate, by a comma, the first three figures*, which constitute *the first period.* Then, proceeding towards the left, *we separate three more*, and so on, regularly, till the whole of the figures are thus separated. As the number may consist of any number of figures, it is evident that the left-hand period will often contain *only one or two figures.*

40. The first, or right-hand period, is called *Units;* the next *Thousands;* the next *Millions;* and so, in succession, *Billions, Trillions, Quadrillions, Quintillions, Sextillions, Septillions, Octillions, Nonillions, Decillions, Undecillions, Duodecillions, Tredecillions, Quatuordecillions, Quindecillions, Sexdecillions, Septendecillions, Octodecillions, Novendecillions, Vigintillions, Viginti-unillions, Viginti-billions*, &c.

41. To facilitate the remembrance of the names of the periods, to show their import and relative position in the scale of numbers, as well as to extend them as far as required by any number within the scope of human thought or calculation, the following Table of Latin Numerals, from which they are nearly all derived, will be found very useful.

From this Table we also derive the names of some months; and there is this singularly coincident irregularity, viz., that as the Romans began their year *in March*, the month *September* was, as its name denotes, in their Calendar, the *Seventh;* whereas, in ours,—to which the name has been transferred, and which begins two months earlier,—it is the *Ninth:* so, also, in the scale of the periods, as we have two names, *Units* and *Thousands*, which do not belong to the regular nomenclature, in which we may suppose that the name *Millions* has been substituted for *Unillions;* the period *Billions*, which, from the import of the word, should be *the second*, is *the fourth*, and, consequently, *Septillions*, which should be *the seventh*, is *the ninth* period.

42. To correct this discrepance, therefore, we must, in applying the Table, *add* 2 *to the number corresponding to the name of the period, (or month,) which will give its order;* or *subtract* 2 *from the number showing its order, which will give the number corresponding to its name.* Thus, for example, if I would know the order of the period *Octillions* or the month *October;* as *octo* is *eight,* I add 2, which makes 10; I therefore say, that *Octillions* is *the tenth period,* and *October, the tenth month.*

43. Again: if I wish to know the name of the *twelfth period,* I subtract 2 from 12, and I have 10; which, in Latin, is *decem:* I therefore say, that the name of the twelfth period is *Decillions.*

Table.

1. Unus.	11. undecim.	21. viginti-unus.
2. duo,(*bis,twice.*)	12. duodecim.	30. triginta.
3. tres, or tria.	13. tredecim.	40. quadraginta.
4. quatuor.	14. quatuordecim.	50. quinquaginta.
5. quinque.	15. quindecim.	60 sexaginta.
6. sex.	16. sedecim.	70. septuaginta.
7. septem.	17. septendecim	80. octoginta.
8. octo.	18. octodecim.	90. nonaginta.
9. novem.	19. novendecim.	99. nonaginta-novem.
10. decem.	20. viginti.	100. centum.

44. By means of the table we may continue the nomenclature of periods, thus: *viginti-trillions,* &c. *to viginti-nonillions; trigintillions; triginta-unillions, triginta-billions,* &c.; *quadragentillions; quadraginta-unillions,* &c.; *quinquagentillions; sexagentillions; septuagentillions; octogentillions; nonagentillions; centillions.*

45. The above Table, as well as the names of the periods in the Scale of Numbers, we could easily extend, but such extension could, for our present purpose, be of no utility; seeing that the number 1 centillion is far above the scope of human affairs or conception. For, if we suppose a hollow globe, *ten millions of millions* (10 *trillions*) of miles in diameter, to be filled with dust, so fine that there should be *one thousand millions* (1 *billion*) of particles in each cubic inch; the whole number of particles in this mighty mass would be an inconceivably small part of a *centillion.*

46. It is plain (see Art. 36) that, whatever may be the value of the units composing any figure in the scale, *the next*

figure on the left will be tens, with regard to that value, and the next hundreds. Wherefore, in each period, going from right to left, we say *units, tens, hundreds:* that is to say, with respect to the digits, any figure on the right is *a number of units;* in the middle, *a number of tens,* and on the left, *a number of hundreds.* Hence, we read the number, under any period, *as if it stood alone,* except merely at last pronouncing the name of the period. Thus, the number 243 is read *two hundred and forty-three,* under whatever period it may stand. For example, in the following number:

M.	Th.	U.
243	243	243

Beginning at the left, we say, *two hundred and forty-three* Millions; *two hundred and forty-three* Thousand; *two hundred and forty-three.*

47. Hence also, *by its place,* we easily determine the value of each figure in a number, without reference to the others. Thus, in the above, beginning at the left, the figure 2 is read, first, *two hundred Millions;* then, *two hundred thousand;* and, lastly, *two hundred.* The figure 4 is, first, *forty Millions;* then, *forty Thousand;* and, lastly, *forty.* The figure 3 is, first, *three Millions;* then, *three Thousand;* and, lastly, *three.*

48. Again: from the same universal law, it is evident that *we can,* in a large number, *read a part, consisting of a number of consecutive figures, taken at pleasure,* (the left-hand figure of such part being significant,) *as a number of units of the order signified by the place of the right-hand figure of the part taken.* Thus, in the number

B.	M.	Th.	U.
128	029	625	343

One hundred and twenty-eight Billions; *twenty-nine* Millions; *six hundred and twenty-five* Thousand, *three hundred and forty-three:* taking the figures *two and two* in order, we read, twelve *tens of Billions;* eighty *hundreds of Millions;* twenty-nine *Millions;* sixty-two *tens of Thousands;* fifty-three *hundreds* and forty-three: or, *four and four,* we read, one thousand two hundred and eighty *hundreds of Millions;* two thousand nine hundred and sixty-two *tens of Thousands;* five thousand three hundred and forty-three: or, taking the two middle figures, we read, ninety-six *hundreds of thousands:* or, taking eight figures, beginning with the figure 8, we read eighty Millions, two hundred and ninety-six Thousand, two hundred and fifty-three *hundreds*, or any other part in the same manner, at pleasure.

49. From the above, we easily perceive that *any number, having a significant figure in the place of units, may be considered as a binomial;* (*bis*, twice, and *nomen*, name;) that is, as consisting of *two names or parts*, one of which is *the number of tens expressed by all the figures on the left of its unit figure;* and the other *the number of units expressed by that figure.* Thus, the above number, considered as a binomial, consists of *twelve Billions, eight hundred and two Millions, nine hundred and sixty-two Thousand, five hundred and thirty-four* tens and *three* units. This general binomial property will be found important in some of our future details.

50. When some of the intermediate orders or periods of a number contain ciphers only, *we omit such orders or periods in reading*, as also *the name Units:* because, *all numbers being composed of units, that name is applicable to any order or period.* Thus, the following number,

B. M. Th. U.
1 000 009 022

is read *One Billion; nine Thousand and twenty-two.*

Let the following numbers be expressed in words:

1. 101
2. 110
3. 111
4. 1001
5. 1010
6. 1100
7. 10010
8. 10002
9. 10120
10. 900090
11. 300400
12. 506011
13. 5006030
14. 40206017
15. 95000070004
16. 694032580795251
17. 875875875875875
18. 12000679800979040
19. 1116171251984235148 91
20. 830065000101100995800004 28

51. When familiar with the Table, and the names derived from it, *we dispense with actually pointing the periods and writing their names.* Thus, in the number

65428075000095008

beginning at the right hand and *taking the figures three at a time, we count the number of periods*, saying, 1, 2, 3, 4, 5, 6, and (42) subtracting 2, we have 4 for remainder, by the Table *quatuor*, and hence, *quadrillions*, for the name of the highest period. Wherefore, having in this the figures 65, we read *sixty-five Quadrillions; four hundred and twenty-eight Trillions; seventy-five Billions; ninety-five Thousand and eight.*

To read this as a binomial, *we begin with the tens in counting the periods;* and finding 6 only in the highest, we say, *six Quadrillions; five hundred and forty-two Trillions; eight hundred and seven Billions; five hundred Millions; nine thousand five hundred* TENS, *and eight* UNITS.

In like manner, let each of the following numbers be read as a Natural and as a Binominal Number:

1. 2986400032504003
2. 1000078540003205
3. 98000043587020050076
4. 6850685006850006850000685

Let the following number be read in parts, taking the figures consecutively two and two, four and four, and five and five, as in Art. 48:

6402640264026402 6402

52. To express, in figures, a large number given in words, *we first write the names of the periods, beginning with Units and ending with the highest or first name mentioned in the reading.* Then, beginning with the left-hand period, we write its number underneath, *as if it were to stand alone.* As we progress from left to right, the order of the places is, under each period, *hundreds, tens, units;* consequently, in writing its number, *we first consider whether it contains hundreds,* and write *a cipher or digit in the left-hand place accordingly.* We next consider *whether it contains tens;* and write, accordingly, *a cipher, or digit in the middle place.* Lastly, in like manner we supply *the place of units.*

If, in the reading, any period is wanting—that is, not mentioned—*we must, of course, write* 000 *under it.*

The cipher is of no use, when placed on the left of all the significant figures; *because, in this situation, it does not show their position towards the left.*

To express, in figures, the number *Forty Quintillions; ten Quadrillions; two Billions; ten Millions; eleven Thousand, four hundred and twelve;* we first write, as above directed, *the names of the periods,*

Quintillions	Quadrillions	Trillions	Billions	Millions	Thousands	Units.
40	010	000	002	010	011	412

Then, as the number first dictates *Forty* Quintillions, we write 40 under Quintillions, *as if it were to stand alone.* Next, for the *ten* Quadrillions, as this number contains *no hundreds,* we first write 0 in the place of hundreds; then, as *ten* is expressed *by a unit in the place of tens,* we write 1 in that place. Lastly, as there *no units,* we write 0 in the place of units: thus, we have 010 (*nought*, *one*, *nought*) under quadrillions. Next, as there are *no Trillions* mentioned in the number, we write 000 (*nought*, *nought*, *nought*) under Trillions. Then, for the *two* Billions, as this number contains *neither hundreds nor tens*, we first write 0 in the place of hundreds; then, 0 in the place of tens; and, lastly, 2 in the place of units: thus we have 002 (*nought*, *nought*, *two*) under Billions. We next write *ten* Millions, as we did *ten* Quadrillions, namely, 010, (*nought*, *one*, *nought*,) under Millions. Then, for *eleven* Thousand, as this is *ten and one*, we first write 0 in the place of hundreds; then 1 in the place of tens; and, lastly, 1 in the place of units: thus, we have 011 (*nought*, *one*, *one*) under thousands. Lastly, we write *four hundred and twelve*, 412, (*four*, *one*, *two*,) under Units, as we should under any other period, namely, *as if it were to stand alone.*

Let the following Numbers be expressed in Figures:

1. *Eighty* Thousand, *six hundred* and *twenty.*
2. *Eleven* Millions; *four hundred* Thousand and *four.*
3. *Fourteen* Millions, *nineteen* Thousand and *eleven.*
4. *Five hundred and four* Millions; *one hundred* Thousand and *one.*
5. *Nine hundred* and *nine* Billions; *thirty* Thousand and *ten.*
6. *Two* Trillions; *three hundred* Billions; *forty* Millions; *four* Thousand.
7. *One hundred* and *ninety* Quadrillions; *twelve* Billions; *one* Thousand.
8. *Fourteen* Nonillions; *sixteen* Sextillions; *three hundred* and *twenty-six* Quintillions; *fourteen* Millions and *six.*
9. *Five hundred* Decillions; *eight* Octillions; *twenty* Septillions; *one hundred* and *one* Quadrillions; *one hundred* and *eleven* Trillions; *nine* Billions; *five hundred* Thousand and *forty-two.*
10. *Ten* Duodecillions; *one hundred* and *nineteen* Nonillions; *fifteen* Octillions; *thirty* Septillions; *four* Quintillions; *five* Trillions; *fifty* Billions; *ninety-one* Millions; *one hundred* and *ten* Thousand and *thirteen.*

53. The Unit, which is repeated or multiplied in forming a number, we shall call the *Genéric Unit*, (*genus*, Lat., *kind*, *sort*,) *because it shows of what kind the number is.* Hence *the abstract unit*, represented by the figure 1, *is the generic unit of all natural whole numbers.* Also, in general, the

unit from which a number derives its name is *the generic unit of that number.*

54. We have seen (49) that a natural number, having its unit figure significant, may be considered *binomial.* This we shall call a *Natural Binomial, (Nat. Bin.)* But a number of consecutive figures, taken at pleasure, is (48 and 53) read *as a natural number, having, for its generic unit, a unit of the order of its right-hand figure:* consequently, if this figure be significant, the number expressed by those figures may also be considered *binomial,* having for its units *the number expressed by its right-hand figure,* and, for its tens, *the number expressed by the figures contained in it on the left of that figure.* This we shall call *a Factitious Binomial, (Fac. Bin.)*

Wherefore, in the same number; though *we can have but one* Natural Binomial, *we may constitute many Factitious ones.*

55. If, on the right of a Nat. Bin., we place a cipher, *it ceases to be binomial,* the whole being now (48) *a number of tens.** It may, however, be considered *a Fac. Bin.,* which may be read *as before,* with this condition, that *the generic unit of each of its parts is ten times as great as before:* hence, *the number is multiplied by ten.*

Again: if another cipher be placed on the right, it is (48) read as a whole, *a number of hundreds:* and as a *Fac. Bin.* the generic unit of each part is 100 *times as great as at first.* Hence *the number is multiplied by* 100.

Thus we see, that to multiply a number by 10, 100, 1000, &c., *we have only to place* 1, 2, 3, &c. *ciphers on the right.*

56. Again: in a number having 0 on the right, the part on the left of this 0, being a number of tens, is *the quotient found in dividing the number by* 10; *because it shows how many times* 10 *can be subtracted.* If there be two ciphers, the part on the left, being hundreds, shows *how many times* 100 *can be subtracted:* and, hence, it is *the quotient in dividing by* 100.

Wherefore, to *cut off* 1, 2, 3, &c., *ciphers on the right* of a number, is to divide that number by 10, 100, 1000, &c.

57. When a unit or number is repeated two or more times, the sum found by adding or combining the whole together is said to be *the product of two numbers multiplied*

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* A number having one name is *Monomial, (monos,* Greek, *one.)*

together or into each other,—one of which is *the number repeated,* or multiplicand, *and* the other, *the number which shows how many times it is repeated,* or multiplier.

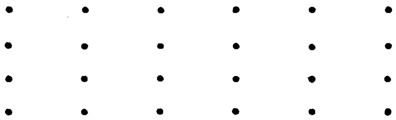
Hence, any number is *the product of its generic unit multiplied by that number.* Thus $1 \times 24 = 24$.

58. If, from a number which contains another a certain number of times, we subtract that other until nothing remains, *the number from which we subtract,* or *dividend,* is said to be divided by *the number subtracted,* or *divisor:* the number of times it is subtracted being *the quotient.*

Hence, any number is *the quotient, found in dividing that number by its generic unit.* Thus $\frac{24}{1} = 24$

Wherefore a number multiplied or divided by its generic unit, remains the same.

59. Of the two numbers forming a product, *either may be the number repeated, or multiplicand; the other being the number of times it is repeated, or multiplier.* For example, if we write separately the units of the figure 6, and repeat the number separated, 4 times, in rows under each other thus:



we see, that for every unit in the row repeated, we have a column containing a number of units, which agrees with the number of times the row is repeated. Thus, for 2 rows, we have 6 columns of 2 units each, or 6 times 2. For 3 rows, 6 times 3. For 4 rows, 6 times 4, as here exemplified, and so on. Hence, we see that the product is formed *by a repetition of either the multiplicand or multiplier,* which two numbers are, therefore, called *the two factors* (*makers, facĕre,* Lat. *to do* or *make*) *of the product;* because *the product is the sum obtained by the addition of the series formed by repeating either factor as often as there is a unit in the other.* Thus, $6 + 6 + 6 + 6 = 4 + 4 + 4 + 4 + 4 + 4 = 24$; or, $6 \times 4 = 4 \times 6 = 24$; that is, 4 times 6 equals 6 times 4: and it is the same with any two numbers multiplied.

60. Again: either factor, in showing how often the other is contained in the product, shows *how often that other could be*

subtracted, and is, consequently, *the quotient* in dividing the product by that other factor. Wherefore, if we divide the product of two numbers by either of them, *the quotient will be the other.* Thus, $\frac{24}{6} = 4$ and $\frac{24}{4} = 6$.

Hence, also, *a number multiplied and divided by the same number is still the same.* Thus, $\frac{6 \times 4}{4} = 6$; that is, *multiplication and division destroy the effect of each other;* also, *addition and subtraction destroy the effect of each other;* which properties are the natural result of *the opposite attributes of quantity.*

SECTION III.

ADDITION.

61. Addition is *the method by which we classify and combine the orders of several numbers so as, of the whole, to form but one number, called the sum.*

Note. The word *Sum* should not be used for *Question.*

By rendering the mind familiar with the combination of each of the digits with each of the others, as in the following

Addition Table:

2 + 2 = 4
2 + 3 = 5
2 + 4 = 6
2 + 5 = 7
2 + 6 = 8
2 + 7 = 9
2 + 8 = 10
2 + 9 = 11

9 + 9 = 18

3 + 3 = 6
3 + 4 = 7
3 + 5 = 8
3 + 6 = 9
3 + 7 = 10
3 + 8 = 11
3 + 9 = 12

8 + 8 = 16
8 + 9 = 17

4 + 4 = 8
4 + 5 = 9
4 + 6 = 10
4 + 7 = 11
4 + 8 = 12
4 + 9 = 13

7 + 7 = 14
7 + 8 = 15
7 + 9 = 16

5 + 5 = 10
5 + 6 = 11
5 + 7 = 12
5 + 8 = 13
5 + 9 = 14

6 + 6 = 12
6 + 7 = 13
6 + 8 = 14
6 + 9 = 15

the addition of the longest line of figures is performed with expedition and certainty. For example, if we know that 8 and 5 is 13, we shall easily perceive that 28 and 5 is 33; that

38 and 5 is 43; that 48 and 5 is 53, &c., seeing that the figure 3 remains the same, and that it is easy to keep account of the additional unit in the place of tens.

62. The correctness of an addition is proved *by adding the figures, a second time, in an inverse or contrary order:* which second addition must, if the work be right, give the same sum as the first; consequently, when a different sum is found, *the work must be repeated till we find the same.* For example, in the following addition,

$$2+3+4+1+5=15$$

beginning at the left, we say, 2, 5, 9, 10, 15: which last we write as the sum. We then begin at the right, and say, 5, 6, 10, 13, 15: which sum, being the same as before, we consider correct.

63. The following Additions are (57) *Multiplications*, and should be remembered for future use. For practice in addition, and to imprint them, as multiplications, on the memory, they may (beginning with 2 and proceeding regularly throughout) be repeated thus: 2 and 2 is 4; twice 2 is 4; 3 and 3 is 6; twice 3 is 6; and (59) inversely, 3 times 2 is 6: and so on for the rest. Again: for the combination of 3 figures, we say, 3 and 3 is 6 and 3 is 9; 3 times 3 is 9: 4 and 4 is 8 and 4 is 12; 3 times 4 is 12; and inversely 4 times 3 is 12, &c. For 4 figures: 4 and 4 is 8, and 4 is 12, and 4 is 16; 4 times 4 is 16: 5 and 5 is 10, and 5 is 15, and 5 is 20; 4 times 5 is 20: also, inversely, (which, throughout the whole, should not be forgotten,) 5 times 4 is 20, &c. In the same manner repeat to 9 figures inclusive.

$2+2=4=2\times 2$
$3+3=6=2\times 3$
$4+4=8=2\times 4$
$5+5=10=2\times 5$
$6+6=12=2\times 6$
$7+7=14=2\times 7$
$8+8=16=2\times 8$
$9+9=18=2\times 9$

$3+3+3=9=3\times 3$
$4+4+4=12=3\times 4$
$5+5+5=15=3\times 5$
$6+6+6=18=3\times 6$
$7+7+7=21=3\times 7$
$8+8+8=24=3\times 8$
$9+9+9=27=3\times 9$

$4+4+4+4=16=4\times 4$
$5+5+5+5=20=4\times 5$
$6+6+6+6=24=4\times 6$
$7+7+7+7=28=4\times 7$
$8+8+8+8=32=4\times 8$
$9+9+9+9=36=4\times 9$

$5+5+5+5+5=25=5\times 5$
$6+6+6+6+6=30=5\times 6$
$7+7+7+7+7=35=5\times 7$
$8+8+8+8+8=40=5\times 8$
$9+9+9+9+9=45=5\times 9$

$$6+6+6+6+6+6=36=6\times 6$$
$$7+7+7+7+7+7=42=6\times 7$$
$$8+8+8+8+8+8=48=6\times 8$$
$$9+9+9+9+9+9=54=6\times 9$$

$$7+7+7+7+7+7+7=49=7\times 7$$
$$8+8+8+8+8+8+8=56=7\times 8$$
$$9+9+9+9+9+9+9=63=7\times 9$$

$$8+8+8+8+8+8+8+8=64=8\times 8$$
$$9+9+9+9+9+9+9+9=72=8\times 9$$

$$9+9+9+9+9+9+9+9+9=81=9\times 9$$

In this last combination, observe, that, as the number 9 requires but one unit to constitute 10, *the unit figure of the number added to it is diminished, and that in the place of tens increased, each by a unit.* Thus: in adding 9 to 9 we have 18; then, in adding 9 to 18, the 8 becomes 1 unit less, and the 1, one unit greater: hence, we have 27; and so, in succession, we have 36; 45; 54; 63; 72; 81.

64. When, in adding, we meet with two digits, the sum of which is 10, we comprehend both at once, *in simply reading our last result, as having one more unit in the place of tens.* Thus, in the following addition,

$$2+3+9+7+3+9+6+4+9+8+2+9=71$$

beginning at the left, we say: 2 and 3 is 5; and 9 is 14; and 10 (taking $7+3$) is 24; and 9 is 33; and 10 ($6+4$) is 43; and 9 is 52; and 10 ($8+2$) is 62; and 9 is 71.

Then, beginning at the right, we say: 9 and 10 is 19; and 9 is 28; and 10 is 38; and 9 is 47; and 10 is 57; and 9 is 66; and 3 is 69; and 2 is 71.

Let the following examples be proved by adding both ways:

1. $2+2+1=5$
2. $1+2+3=6$
3. $4+1+2=7$
4. $2+2+4=8$
5. $3+4+2=9$
6. $3+5+2=10$
7. $4+5+2=11$
8. $6+4+2=12$
9. $8+2+3=13$
10. $8+3+3=14$
11. $7+3+5=15$
12. $6+5+5=16$
13. $8+4+5=17$
14. $9+4+6=19$
15. $9+5+6=20$
16. $9+7+6=22$
17. $9+7+8=24$
18. $9+8+8=25$
19. $9+9+8=26$
20. $9+9+9=27$

Let the results of the following be written and proved, as in the foregoing:

1. $5+4+4+3=$
2. $5+6+5+6=$
3. $7+4+3+8=$
4. $8+5+8+6=$
5. $8+6+8+9=$
6. $9+7+8+8=$
7. $7+9+5+6=$
8. $4+9+9+3+8=$
9. $5+6+8+9+7=$
10. $7+4+9+8+8=$
11. $7+3+5+8+9+6=$
12. $5+6+8+7+8+9=$
13. $9+9+5+7+4+7+3=$
14. $6+6+9+7+9+5+6=$
15. $4+8+6+3+9+7+8+6=$
16. $3+9+7+5+8+8+9+5=$
17. $5+5+8+3+7+5+9+6+9=$
18. $8+7+6+4+7+3+8+2+9=$
19. $9+7+3+8+5+2+5+7+9=$
20. $9+9+8+8+4+7+6+5+8=$

65. All the units contained in many given finite or limited numbers, however great, can be expressed *by a single finite number.* For, as the Scale of Natural Numbers is (26) infinite, progressing by a unit at a time, it is evident that in continuing its formation, we shall, at length, arrive at a number expressing all the units contained in the given numbers.

66. Some learners perform the addition of large numbers, by separating the units composing each figure in those numbers and adding them one at a time, which slow enumeration very much retards their progress. The necessity for this injurious practice may be obviated *by a frequent repetition of the addition table.*

67. From the above, it is plain that the only difference between the formation of the sum of an addition and the regular formation of numbers, as in numeration, is, that in adding, *instead of progressing by a unit at a time, we comprehend, at each step, all the units contained in any digit which may present itself to our view.*

68. In forming the sum of large numbers, care must be taken *to combine with each other only those figures which express units of the same order.* For, as it is by ten units of the same order that we form one unit of the next order towards the left, it is plain that, *if we add units of different orders to each other, the sum, not being of any order, cannot be expressed,* neither can it form new orders.

69. *Because it is with a unit that we begin to form numbers, and proceed towards the left,* in adding large numbers, we must begin by adding the unit figures, then the tens, and so on.

70. Also, *because ten units of any order are expressed by one unit of the next order on the left,* we carry one to the next order for every ten which we find in the sum of any order of units.

71. When the sum of any order is an exact number of tens; that is, when its unit figure is 0, this must be written underneath the order, and the tens—which (49) comprise all the other figures—must be carried to the next order. If we omit this cipher, *the digits on the left will* (56) *be ten times less than they should be.*

Examples.

1. $453 + 95 + 776 + 81 = 1405.$

We begin on the right, and first add the unit figures, saying: 1 and 6 is 7, and 5 is 12 and 3 is 15. Then, taking room on the right of the sign $=$, as 15 is ten and 5, we write 5, and carrying 1, which is ten, to the place of tens, we add the figures in that place, saying: 1 and 8 is 9; and 7 is 16; and 9 is 25; and 5 is 30. Now, as this is 30 tens, or 3 hundreds, we place 0 on the left of 5; and carrying 3, we add the hundreds, saying: 3 and 7 is 10, and 4 is 14; which we write on the left of 05, and have 1405 for the sum. The figure 1, which is ten hundreds, being, in accordance with the general law, in the units place of thousands.

To prove the work, we begin on the left, and add thus: 3 and 5 is 8; and 6 is 14; and 1 is 15. We write 5, saying 5 and go 1 to 5 is 6; and 9 is 15; and 7 is 22; and 8 is 30; 0 and go 3 to 4 is 7; and 7 is 14.

Let the following numbers be added in the position in which they stand, and the work proved as above:

2. $34 + 151 + 987 + 72 =$
3. $2054 + 15 + 876 + 47 =$
4. $51573 + 358 + 9256 + 19 =$
5. $92 + 1509 + 435 + 79847 + 5840 =$
6. $317 + 25 + 8040 + 92273 + 38029 =$

72. When the numbers to be added are not only large, but numerous, it is usual, *for the sake of convenience,* to place them under each other, so that units of the same order may stand in the same column. This is easily done, *in placing*

the unit figures under each other, and the other figures in regular order towards the left; whereby, as all take their value by the same general law, from their position, as they advance in this direction, each figure will be found in a column with figures of the same order of units—that is, *if units be under units, tens will be under tens, hundreds under hundreds,* &c. We then begin *with the units,* and add as usual; taking care, after writing the unit figure of the sum of each column, *to write also, underneath this figure, the tens which are to be carried to the next column.* This is done *for the sake of reference,* and is very useful in long additions; for if, while adding, any thing should call off the attention, we are not obliged to resume the work from the beginning.

In writing the numbers of the following examples, to facilitate the work, we first write under each other those of the lowest order; then those of the next higher, and so on till all are written.

Example.

To add the numbers four hundred and ninety-six; three thousand seven hundred and forty-nine; nine hundred and thirty-seven; seventeen thousand and eighty-five; five millions nine hundred and eighty-eight thousand four hundred and seventy-three; we place them thus:

```
M. Th. U.
      496
      937
     3749
    17085
  5988473
  -------
  6010740
   112233
```

Then, beginning at the right-hand column, (for the sake of greater despatch, *we omit all intermediate language,*) and say: three, eight, seventeen, twenty-four, thirty. We write 0 under the column of units. Under this 0 we also write the 3 tens, which we carry to the next column, and say: ten, eighteen, twenty-two, twenty-five, thirty-four. Now, as this is 34 tens, or 3 hundreds and 4 tens, we write 4 under tens, and under this 4 the 3 hundreds, which we carry to the next column. In the same manner we proceed with the other columns till the operation is finished.

To prove the work, *we begin at the top of each column and add downwards,* taking them in order from right to left.

73. The scholar, having acquired some facility in adding, should *entirely drop the use of intermediate language, except in carrying from one column to another,* as in the following example:

$$\begin{array}{r} 6935 \\ 4872 \\ 5888 \\ 5975 \\ 7978 \\ \hline 31648 \\ 432 \end{array}$$

where we say simply, 8, 13, 23, 28; 8 and go 2 to 7 is 9, 16, 24, 34; 4 and go 3 to 9 is 12, 21, 29, 37, 46; 6 and go 4 to 7 is 11, 21, 31, which last we set down in full.

3. Add together three hundred and thirty-six; ninety-seven; three thousand and seventy-four; eight hundred and eighty-seven; and six hundred and forty-three thousand eight hundred and nine.

Answer. Six hundred and forty-eight thousand two hundred and three.

4. Required the sum of eighty-two; four hundred and seventeen; forty-nine; three thousand seven hundred and thirty-eight; and eleven millions five hundred thousand and twenty-four.

Ans. Eleven millions five hundred and four thousand three hundred and ten.

5. Required the sum of seven thousand two hundred and nineteen; sixty-three; five hundred and sixty-four; fourteen millions seven hundred and nine thousand and forty-five; twenty-seven thousand nine hundred and ninety; seventy-seven; nine thousand and ninety-seven; and ninety millions thirty-three thousand six hundred.

Ans. One hundred and four millions seven hundred and eighty-seven thousand six hundred and fifty-five.

6. Required the sum of two thousand three hundred and fifty-six; sixty-six; three hundred and nineteen; twelve millions five hundred and seven thousand and twenty-six; ninety-four; seven hundred thousand and fifty-nine; four thousand

six hundred and seventy-four; and nine hundred and eight millions and six.

Ans. Nine hundred and twenty-one millions two hundred and fourteen thousand six hundred.

7. Required the sum of thirty-seven; six hundred and fifty-five; three thousand and seventy; twenty millions ninety-four thousand and fifty-seven; seventy-four thousand and forty-seven; six thousand and ninety-nine; three hundred and twenty-eight; fifty-seven thousand five hundred and six; and six hundred and six millions five hundred thousand seven hundred.

Ans. Six hundred and twenty-six millions seven hundred and thirty-six thousand four hundred and ninety-nine.

8. What is the sum of fifty-eight; three thousand and four; eighty-three; ten thousand four hundred and ninety; three hundred and fourteen; six millions six hundred and four thousand and seventy-five; two hundred and twenty-four; eleven thousand and fifty; and five billions one hundred and four thousand and seventy-two.

Ans. Five billions six millions seven hundred and thirty-three thousand three hundred and seventy.

Let the following examples be performed and proved, as above:

123	426	627	713
316	321	832	428
713	242	415	596
332	651	352	362
244	344	486	485
352	163	379	764
614	523	861	819
221	333	943	926

542	4732	595	999999
762	3567	667	526483
566	4536	649	384625
865	2246	887	721983
535	7267	556	389127
232	4532	856	623915
432	5463	997	519326
464	7753	878	224179
442	3464	884	971422

7537
8945
3793
9596
6567
8549
2468
3978
8876
6947
6924
4359
8977

8658
4956
8524
9987
6593
8648
6327
8398
7892
5647
7256
4967
7979

9784
9639
7964
9625
8759
8965
9345
9657
3584
4559
8245
8659
3878

7956
3614
5728
3245
6172
8518
2536
4544
6385
4271
6754
3857
1481
7463
5455

8469
2572
8688
9247
3584
5763
2538
9647
3457
7784
3569
4937
7865
8526
3444

7776
5842
3757
2986
5339
9987
2854
6947
5576
9669
7489
8546
7578
6259
9394

74. The unit figure of the sum of any column, being of the order of the column, is written underneath; and as the other figures of this sum express (48) a number of tens with regard to that figure, they are carried, as that number, to the next column on the left; the figures of which, (36) in the same sense, are also tens. Therefore, *when the sum of a column equals or exceeds* 100, *the number to be carried will be* 10 *or more: when the sum equals or exceeds* 200, *the number to be carried will be* 20 *or more: when the sum equals or exceeds* 300, *the number to be carried will be* 30 *or more*, &c. On reaching 100, the scholar may make a dash at the side of his work, and proceed with the surplus, as at first.

5397	6827	7878	9247
8423	5473	6945	8939
9467	6227	9543	4964
8528	8294	8773	9789
9726	3764	9685	2576
9453	7753	7552	8437
8597	7289	8788	5865
4867	9936	9722	5895
5453	5979	5896	7788
2847	9374	6539	9927
9993	8495	5789	8534
5728	8739	8466	6782
6958	5687	8837	9549
7273	8547	4983	6989
8576	9778	9956	2774
2398	9735	7454	4986
7597	6849	8793	3927
6385	7744	9749	7788
5255	8392	6648	5947
3446	9969	8658	9969
8976	9889	9776	2879
5843	3897	9437	3729
7555	5955	9586	9569

Addition may be considered the principal, and, indeed, the most laborious operation of the Counting-house. The pupil, therefore, should, by all means, endeavour to render himself as skilful as possible in this most important operation. The old adage—"practice makes perfect"—is certainly applicable here, if anywhere; and there can be little doubt that one of the very best methods to adopt is to repeat the same operation many times, writing the result each time, by which the student will not only clearly perceive his liability to err, but also the progress he is hourly making towards perfection. He should also frequently repeat each operation by taking the figures in an inverted or contrary order.

The answers to these questions are omitted, because the insertion of them might induce carelessness in the scholar, and because the teacher, having once seen them, will, by a mere glance of the eye, determine whether the work offered for inspection is correct or not; and the scholar should not present his work till he has found the same result by two consecutive additions performed in a contrary order.

9785	73466	39624	52567
8993	58529	57863	89237
6769	47988	85647	45678
2856	53547	92958	92846
5537	88622	83459	56375
4689	33664	27766	28426
3898	82573	88342	88777
7965	77648	27923	33637
8796	92595	99424	89975
9365	74929	55277	49638
5597	44972	83922	27946
4468	62833	95986	83577
6257	25297	24654	25842
3889	34569	83756	89764
7989	87644	77732	55844
6985	55827	83988	97358
7397	33947	65689	64643
8836	62352	49786	77396
9267	86579	32246	99884
2745	23745	85698	59847
3498	87966	97235	32376
9678	25348	44468	99333
6554	85462	52392	48877
4367	92999	84675	55634
5879	55774	39375	84424
7983	86258	47892	58957
8235	23644	35673	89752
9396	37337	43939	25846
9927	24685	56599	39575
7869	94927	99877	88647

SECTION IV.

SUBTRACTION.

75. THE word *Subtraction* is derived from the Latin words *sub*, out or away, and *trahĕre*, to draw, to take; and, therefore, signifies *to draw out* or *take away*.

76. There can be no difference between two equals. Also, if the whole of a thing be taken away, there can be no remain-

der. Subtraction, therefore, as an arithmetical operation, is *the method of finding the difference between two unequal numbers, by arranging their orders so as to diminish the greater by as many units as are contained in the less.* The remaining part of the greater number is, consequently, called *remainder* or *difference.*

77. The greater number, then, is composed *of the remainder and the less number;* consequently, to prove subtraction, *we add the remainder and less number together,* the sum of which equals the greater, when the work is right.

78. If we take either of two numbers from their sum, the remainder will be *the other.* Wherefore, when the work is right, if we subtract the remainder from the greater number, we shall obtain *the less.*

79. Of two unequal numbers, let 19 be the greater and 10 the less. From 19, to subtract 10, we take away the unit 1, which is ten, and the remainder is 9. If to this remainder 9 we add the less number 10, we have the greater number 19: and this is done by replacing the unit on the left of 9. Again: if from 19 we take the remainder 9, we have the less number 10.

Hence, we see that to subtract 10 from any of the numbers 11, 12, 13, &c., 19, is merely *to take away the unit in the place of tens.* Also, to add 10 to any of the digits 1, 2, 3, &c., 9, is merely *to place a unit on the left of it.*

80. As the sign minus — represents a simple subtraction, so the sign ÷ represents a continual subtraction, which divides the number from which we subtract, into equal parts. In both cases—that is, when either of these signs is placed between two numbers—it is always *the number on the right of the sign* which is to be subtracted from that on the left. Thus, $9 - 3 = 6$. Here, we say: 3 from 9, six; and, inversely, (78,) 6 from 9, three. Again: $9 \div 3$ or $\frac{9}{3} = 9 - 3 - 3$, &c., as often as we can. Thus, $9 - 3 = 6$; $6 - 3 = 3$, and $3 - 3 = 0$. Where, having made three subtractions and arrived at 0, we say that 9 contains 3 just 3 times; that is, when 9 is divided into equal parts of 3 units each, the three subtractions show there are 3 of those parts; and this number 3 is thence called *the quotient,* (*quŏtiĕs,* Lat., *how often?*)

81. The frequent repetition or conning of the following Table will greatly facilitate the progress of the learner. In

each equation, where the remainder and less number are not alike, first subtract the less number and afterwards the remainder. Thus, 2 from 5, three; 3 from 5, two; 2 from 6, four; 4 from 6, two; 2 from 7, five; 5 from 7, two, &c., throughout. This will make the Table, as a preparatory exercise, more complete.

Subtraction Table.

$4-2=2$
$5-2=3$
$6-2=4$
$7-2=5$
$8-2=6$
$9-2=7$
$10-2=8$
$11-2=9$

$18-9=9$

$6-3=3$
$7-3=4$
$8-3=5$
$9-3=6$
$10-3=7$
$11-3=8$
$12-3=9$

$16-8=8$
$17-8=9$

$8-4=4$
$9-4=5$
$10-4=6$
$11-4=7$
$12-4=8$
$13-4=9$

$14-7=7$
$15-7=8$
$16-7=9$

$10-5=5$
$11-5=6$
$12-5=7$
$13-5=8$
$14-5=9$

$12-6=6$
$13-6=7$
$14-6=8$
$15-6=9$

82. Of the two unequal numbers, whose difference is required, the less may be *any number in the natural scale, from* 1, *to the greater minus* 1. Now, if the numbers consist of several orders, and we require the subtraction of a unit of any order towards the left,—as the value of this (37) is greater than that of all the figures which could stand on the right of it, and (46) ten times less than the unit of the lowest order on the left, *the subtraction of it would be impossible in any order, other than that to which it belongs.* Hence the necessity, in arranging the numbers, for placing *units under units, tens under tens,* &c., which is done, as in Addition, *by placing the units under units, and the other figures in regular order towards the left.*

83. The numbers 10, 100, 1000, &c., being (37) greater than 9, 99, 999, &c. &c., it is evident that in arranging, as we do, the less number under the greater, so that units may be under units, tens under tens, &c., *all the figures of the less number may be greater than the corresponding figures of the greater.* Also that *all the orders of the greater, from which we are required to subtract, may be wanting.* But, in either case, the greater number must have, *at least, one unit of an order on the left of the highest order of the less number;* and, as this (37) is greater than all the figures which can stand on the right of it,—and, consequently, than the whole of the less

number,—*we can, from the value of this unit, form all the orders which are wanting in the greater number, and subtract those of the less number from them.*

84. If, to each of two unequal numbers, we add the same number, *this addition has no effect upon their difference.* Because, it is evident that, by adding to the greater number, *we increase the difference;* and by adding to the less, *we diminish it.* Wherefore, as these two opposites (60) destroy the effect of each other, *the difference is still the same.*

85. When the lower figure is greater than the one above it, and, consequently, its subtraction impossible, *we add ten to the upper figure;* that is, (79,) *we read it as if it had a unit on the left.* From this we subtract the lower figure, and write the remainder underneath. Then, proceeding towards the left, we add a unit to the next lower figure; or, rather, *we read it one unit greater than it is,* and subtract as before. By this process, *the same number is added to each of the given numbers;* because (36) *the ten which we add to the upper figure, is equal to the unit which we add to the next lower one;* consequently, *the difference* (84) *is not affected.*

Examples.

1. 4520376 — 831497 = 3688879

$$\begin{array}{rl} 4520376 & \text{Greater.} \\ 831497 & \text{Less.} \\ \hline 3688879 & \text{Remainder.} \\ \hline \end{array}$$

$$\begin{array}{lrl} \text{Proof by Addition,} & 4520376 & \text{Gr. (sum of Rem. and Less.)} \end{array}$$

In this example, the lower figures being all greater than those above them, we add ten to each of the upper figures, except 4. We, therefore, read them in succession, beginning at the right, 16, 17, 13, 10, 12, 15. Also, in consequence, we add one to each of the lower figures 9, 4, 1, 3, 8, which we read successively, always going from right to left, 10, 5, 2, 4, 9.

The enunciation, then, is: 7 from 16, *nine;* 10 from 17, *seven;* 5 from 13, *eight;* 2 from 10, *eight;* 4 from 12, *eight;* 9 from 15, *six;* lastly, as there is no figure under 4, 1 from 4, *three.*

86. When some of the lower figures are greater than those above them, and some less, *we must not add a unit to the next lowest figure, when we have not added ten to the preceding*

upper one. The following example will assist the learner in guarding against this error:

2. 320571086 — 109720158 = 210850928
320571086 Greater.
109720158 Less.
210850928 Remainder.
Proof by Addition, 320571086 Greater.
Proof by Subtraction, 109720158 Less.

Here, we say: 8 from 16, *eight;* 6 from 8, *two;* 1 from 10, *nine;* 1 from 1, *nought;* 2 from 7, *five;* 7 from 15, *eight;* 10 from 10, *nought;* 1 from 2, *one;* 1 from 3, *two.*

Observe, that *it is of little consequence* whether the less number is placed under the greater, or the greater under the less. For, having proved the above example by addition, in order to prove it by subtraction, *we perform the operation downwards:* that is, we subtract the remainder from the number below it, saying, 8 from 16, *eight;* 3 from 8, *five;* 9 from 10, *one;* 1 from 1, *nought;* 5 from 7, *two;* 8 from 15, *seven;* 1 from 10, *nine;* 2 from 2, *nought;* 2 from 3, *one.*

Let all the subsequent examples in subtraction be proved both by addition and subtraction. Also, let the numbers be written in the form of those in the foregoing example.

3. From four hundred thousand and sixty-two, take ninety thousand eight hundred and seven.

4. From eight millions, three hundred and seventy thousand and sixty, take five millions, nine hundred and forty-three thousand and seven.

5. From nine millions, five hundred and ten thousand five hundred and eighty-seven, take six millions, eight hundred thousand seven hundred and sixty-nine.

6. From one hundred millions, one hundred and eight thousand and fifty-eight, take ninety-nine millions, seventeen thousand three hundred and forty-nine.

7. From eighty-three millions, four hundred and one thousand two hundred and sixty-three, take four millions, three hundred and one thousand eight hundred and twenty-four.

8. From sixteen millions, one hundred and thirty-nine thousand seven hundred and twenty-four, take seven millions, two hundred and nine thousand eight hundred and fifteen.

9. From forty-three millions, one hundred and ten thou-

sand eight hundred and thirty-two, take eight millions, four hundred thousand eight hundred and fifty-three.

10. From ten millions, take ninety thousand and seven.

11. From three hundred and two millions, one hundred and eighty-seven thousand two hundred and ninety-one, take nine millions, ninety-six thousand three hundred and ninety-two.

12. From one billion and forty-nine, take nine millions, ninety thousand and ninety-nine.

13. 9990003814 — 9890004905 =
14. 30201150839 — 9101240095 =
15. 64501100123 — 64401100124 =
16. 2000003984565 — 1999304859739 =
17. 1000009017834 — 9809908769 =
18. 7893011053241 — 1795050072937 =
19. 600001238497 — 21858788 =
20. 1000001799112 — 999999998107 =

87. A unit followed by ciphers we shall call *an alt unit,* (*altus,* Lat., *high.*) Thus, 10 is the *first alt,* (1 *alt;*) 100, (2 *alt;*) 1000, (3 *alt,*) &c. The number which precedes the word alt shows the number of ciphers on the right of the unit.

The first, second, third, fourth, &c. alt, we shall call *the alt of any number consisting of* 1, 2, 3, 4, &c. *figures respectively.* But (37) these alts are expressed by the binomials 9 + 1; 99 + 1; 999 + 1; 9999 + 1, &c.; that is, *the number of nines* in the bin. value of an alt, *corresponds with the number of figures* in the number of which it is the alt.

88. *The difference between a number and its alt* is called its *arithmetical complement,* (*ar. comp.*) Therefore, when the sum of two numbers is an alt unit; *each number is the ar. comp. of the other.*

89. To find the ar. comp. of a number, (87,) *we begin at the left, and subtract each of its figures, successively, from* 9; *taking care to subtract the right-hand figure from* 10, which, being 9 + 1, completes the bin. value of the alt. Thus: to find the ar. comp. of 23564, beginning with 2, we say: 7, 6, 4, 3, 6: consequently, the complement is 76436.

The numbers 23564 and 76436, which make the 5 alt, are the ar. complements of each other.

This is so easy, that the scholar will, by dividing any ordinary number into periods, see its ar. comp., as it were, intuitively, and read it with nearly the same facility as the number itself. Thus, if we have the number 26543257, dividing it, by the eye merely, we read at once its ar. comp., which is, *seventy-three* millions; *four hundred and fifty-six* thousand, *seven hundred and forty-three.*

90. If to a number A, we add the ar. comp. of a less number B,—as this ar. comp. (88) is the alt of B — B, we evidently increase A by this alt and diminish it by B. If, then, from the result, we subtract the alt, we have the difference between A and B. Thus, to find A — B,

Let A = 92245812 and B = 73456743
Then, A = 92245812
ar. comp. B = 26543257 = 100 millions — B or 8 alt — B.
$\cancel{1}$ 18789069 Sum or A + 8 alt — B.

Here, suppressing the + 8 alt, (100 millions,) by which the result is too great, we have A — B = 18789069

Proof. From 92245812 subtract 73456743
92245812 Gr.
73456743 Less.
18789069 Rem.

Wherefore, to find the difference between two unequal numbers, we have the following rule: *To the greater add the ar. comp. of the less, and, from the result, subtract a unit of the order of the alt of the less.*

91. In finding by the ar. comp. the diff. between A and B, or (A — B,) when the alt of B is within the compass of A,—that is, when A has more figures than B,—we may, in writing A, diminish it by a unit of the order of this alt; which is the same as to diminish the result, and more convenient. Should this order, or two, three, &c. consecutive orders, as we proceed to the left, be wanting in A, *write* 9 *in each vacant order, and diminish the next digit towards the left by a unit.* Thus:

Let A = 56000902346 and B = 9903077
A — 7 alt = 55990902346
ar. comp. B = 96923
55990999269 diff. or (A — B).

Here, as the alt of B is the 7 alt, or 10 millions, and, as this order and the next on the left are both wanting in A, we write 9 in each of these two places, and diminish the next figure 6 on the left by a unit; by which means (48) we have 599 units of this order instead of 600.

$$\begin{array}{rr} \text{Proof. } A = & 56000902346 \\ B = & 9903077 \\ \hline & 55990999269 \end{array} \text{ (A — B) as before.}$$

Let the following examples be performed and proved in like manner:

1. $997325 - 987406 =$
2. $520463 - 229074 =$
3. $1192076 - 999249 =$
4. $90000724 - 999637 =$
5. $10000085060 - 9988204 =$

92. When B is a series of nines,—as the ar. comp. is, in this case always 1,—*we have only to diminish* A *by the alt and add* 1 *to its unit figure.*

1. $4250484 - 999999 = 3250485$

Here, as the alt is 6 alt, we merely subtract 1 from the 4 on the left and add 1 to that on the right.

2. $80524631 - 9999999 =$
3. $53072058 - 99999 =$
4. $7000092406 - 9999999 =$

93. A series of nines is the alt minus 1. Therefore, to add a series of nines to a number, is *to increase it by the alt and diminish it by a single unit.* Thus:

1. $7854 + 999 = 8853$
2. $7854 + 9999 = 17853$
3. $7854 + 99999 = 107853$
4. $7854 + 999999 = 1007853$
5. $9997854 + 99999 = 10097853$

94. The signs + and —, symbols of the opposite attributes of quantity, are further distinguished by the following appellatives: The sign +, which shows the increase of quantity,—the operation by which it is numerically formed, and, hence, may be said to affirm its positive or real existence,—is called *the affirmative or positive sign:* and the sign —, the symbol of the operation by which it is diminished and ultimately destroyed, and which may, therefore, be said to deny it existence, is called *the negative sign.*

95. A number, having + on the left of it, is called *a positive number,* (real number.) A number, having — on the

left, is called *a negative number*, (imaginary number;) because, of itself, it expresses a want of the number of units, to render it equal to 0, which as a real number it would express. Thus, $-5+5=0$. Hence, *the positive and negative value of the same quantity destroy each other.* Numbers without sign are, of course, *positive.*

Note. The expression on the left of the sign $=$ is called *the first member*, and that on the right, *the second member* of the equation.

96. Let a be the positive, and $-a$ the negative value of any quantity or number: also, let $d=$ the difference between the two values. Then, if to each value we add a, this (84) will not affect d. Wherefore d is the difference between $a+a$ and $a-a$: but $a-a=0$, (95;) therefore, d is the diff. between $a+a$ or $2a$ and 0; consequently, $d=2a$; that is, the difference between the positive and negative value of a quantity is *twice its positive value, or the double of that quantity.*

Hence, *to subtract the negative value of a quantity is the same as to add its positive value.*

97. This has sometimes been familiarly illustrated thus: There are two persons, A and B, one of whom, A, has five dollars, ($+5$;) the other, B, has no money, and is in debt five dollars, (-5.) Now, to discharge his debt—that is, to render him even with the world, or having nothing, (0,) B would require five dollars, and, to be on a par with A, five more: therefore, A is 10 dollars better off than B: that is, the diff. between $+5$ and -5 is 10. Also, as the cancelling or taking away of the debt of B is the same as to give B five dollars, we say, that to subtract -5 is the same as to add $+5$. Therefore, *when several negative quantities are to be subtracted, they must be added as positive quantities*—that is, *all their signs must be changed from minus to plus.*

98. The scholar must not confound the idea of the subtraction of negative quantities with that of their addition. Thus, $a-a$ is the addition of the negative value of a to its positive value; or, which is the same thing, *the subtraction from* a *of its positive value.* The expression of the subtraction of its negative value, without changing its sign, would stand thus: $a-(-a,)$ which looks awkward, and is much better written $a+a$.

Wherefore, (as negatives, or quantity destroyers,) the writing of negative quantities, in any expression, with their appropriate sign $-$, implies *their addition;* with a contrary sign, *their subtraction.*

99. To find the value of an expression in which there are several positive and several negative numbers, *add all the positive numbers together: then add all the negative numbers together, the difference between the two sums, preceded by the sign of the greater, is the value of the expression.*

1. $569 + 19 - 34 + 17 - 413 - 9 = 149$

569	-34
19	413
17	9
605 Sum of positive.	-456 Sum of negative.

Then $605 - 456 = 149$. The greater sum being positive, *the remainder* 149, which is the value of the expression, *is positive,* and is placed on the right of the sign $=$.

Or, as it is of no consequence in what order the numbers are placed, we may proceed thus:

$$569+19-34+17-413-9=569+19+17-34-413-9$$
$$=605-456=149$$

In this case the work may be performed without placing the numbers under each other.

2. $999 - 2014 + 164 - 13 - 79 + 40 = -903$

999	-2014	
164	13	-2106 Sum of neg. num.
40	79	1203 Sum of pos. num.
1203	-2106	$-$ 903 diff. or required val.

In this example, the sum of the negative numbers being the greater, *the value of the expression is negative.*

3. $254 + 649 - 46 - 75 + 843 - 997 =$
4. $327 + 542 - 39 + 89 - 903 - 14 =$
5. $1050 - 5198 + 23 - 14 + 7 - 120 =$
6. $303 - 27 + 124 + 500 - 809 - 99 =$

100. The ar. comp. may here be of great use. Ascertain by the eye whether the result is likely to be positive or negative, writing the ar. comps. of the numbers of contrary sign instead of the number: also taking care (90) to diminish by e alt unit for every complement used.

1. 5846 + 2359 — 426851 — 4567 = —423213
—406851
4567
4154 ar. comp. of 5846
7641 ar. comp. of 2359
— 423213

Here, as we easily see that the result will be negative, we write the ar. comps. of the two positive numbers, the alts of which being 4th alt, we diminish the largest negative number by 2 units of this order.

The scholar may prove the work by pursuing the ordinary method:

2. 532946 — 171 — 3436 + 546 — 27 = 529858
521846
829 ar. comp. of 171
6564 ar. comp. of 3436
546
73 ar. comp. of 27
529858

The result being evidently positive, we take the ar. comp. of the three negative numbers: and, as the alts are 2d, 3d, and 4th, we diminish the largest positive number by 1 unit of each of these orders.

The scholar may prove as before.

3. 54927 — 9999 + 99 — 9 =
4. 483216 — 45 — 259 + 354 =
5. 5436729 — 29346 + 9999 =
6. 64837 — 99 + 999 — 7254 =
7. 999 + 58476 — 356742 — 187 =
8. 9473 — 9999 — 549 + 64 + 999 =
9. 624 — 569 + 1940 — 2835 + 943 =
10. 854327 — 3745 + 9254 — 473 — 49 =
11. 747599 — 274 + 9999 + 999 — 8451 =
747597 + 726 + 1549 = 749872 (90 and 93)
12. — 43560 — 999 — 99999 + 547 + 57 + 9999 =

SECTION V.

MULTIPLICATION.

101. The word multiplication is derived from two Latin words—*multus*, many, and *plicare*, to fold.

As an operation, it is *the method of finding, with greater facility than by either Numeration or Addition, the sum produced by the combination of all the units expressed by writing a number two or more times.* (See Art. 59.)

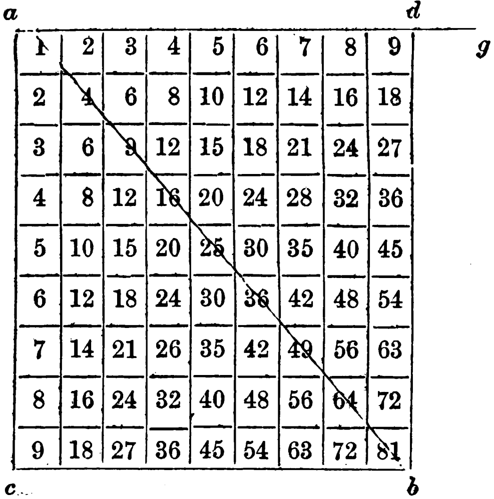
102. The multiplication of numbers, however great, requires *nothing more than the multiplication of a single figure by a single figure.* Hence, the following Table, which contains all the products arising from the multiplication of any two of the figures 2, 3, 4, 5, 6, 7, 8, 9, will prepare the scholar to perform any multiplication whatever. To make the exercise complete, he must perform each multiplication, where the factors are not alike, both ways. Thus, for the expression $2 \times 3 = 6$, he must say, *twice* 3 *is* 6; *three times* 2 *is* 6. For the expression $2 \times 4 = 8$, he must say, *twice* 4 *is* 8; *four times* 2 *is* 8, and so on.

Multiplication Table.

$2 \times 2 = 4$	$3 \times 3 = 9$	$4 \times 4 = 16$	$5 \times 5 = 25$
$2 \times 3 = 6$	$3 \times 4 = 12$	$4 \times 5 = 20$	$5 \times 6 = 30$
$2 \times 4 = 8$	$3 \times 5 = 15$	$4 \times 6 = 24$	$5 \times 7 = 35$
$2 \times 5 = 10$	$3 \times 6 = 18$	$4 \times 7 = 28$	$5 \times 8 = 40$
$2 \times 6 = 12$	$3 \times 7 = 21$	$4 \times 8 = 32$	$5 \times 9 = 45$
$2 \times 7 = 14$	$3 \times 8 = 24$	$4 \times 9 = 36$	
$2 \times 8 = 16$	$3 \times 9 = 27$		$6 \times 6 = 36$
$2 \times 9 = 18$		$7 \times 7 = 49$	$6 \times 7 = 42$
	$8 \times 8 = 64$	$7 \times 8 = 56$	$6 \times 8 = 48$
$9 \times 9 = 81$	$8 \times 9 = 72$	$7 \times 9 = 63$	$6 \times 9 = 54$

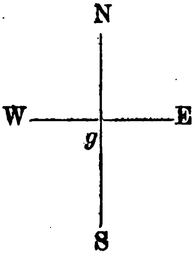
103. The numbers produced by the continual addition of each of the digits form the following Table, which is attributed *to Pythagoras.* The first line is formed *by adding* 1 *to itself*, and continuing the addition till we have 9 times 1. The second line, *by adding* 2 in the same manner. The third, *by adding* 3, *and so on for the others.*

Pythagorean Table.



1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	26	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

When two straight lines cut each other, so as to make the four angles at the point of intersection all equal, the lines are said to be *perpendicular to each other*, and the angles are called *right angles*. Thus, the four angles, which the straight lines N S and W E make at the point *g*, being equal, are right angles, and the lines are perpendicular to each other.



104. The surface or outside of any thing is called *its superficies*, which is considered as having only *length and breadth*. A *plane* is that superficies in which, if any two points be taken, *the straight line between them lies wholly in that superficies.*

105. A geometrical figure is the space enclosed by one or more boundaries. The quantity of space enclosed is called *the area of the figure.* Figures enclosed by straight lines are called *rectilineal figures:* if by three straight lines, *trilateral figures or triangles:* if by four, *quadrilateral:* if by many, *multilateral figures or polygons.*

106. *A square* is a quadrilateral figure, having all its sides equal and all its angles right angles.

A right-angled parallelogram, which is also called *a rectangle,* is a quadrilateral figure, having all its angles right angles, but only its opposite sides equal.

107. The Pythagorean Table, which has all its sides equal, and all its angles right angles, is a square. As the divisions of its sides are also equal, suppose each to be 1 inch long: then, as the small spaces in which the numbers are placed are formed by straight lines, one inch apart, which cut each other at right angles, they are all square inches. Now, as each row contains 9 square inches, the figure 9 in the upper row, on the right, shows *the area of the rectangle formed by that row:* and it is plain, that as we descend in the right-hand column, we shall have as many times 9 as there are units in the number in the left-hand column, (59;) that is, the number on the right expresses *the area of a parallelogram, consisting of as many rows of* 9 *square inches as there are units in the left-hand figure;* which figure is the breadth of the parallelogram. Wherefore, to find the area of any parallelogram, *we multiply its length by its breadth,* referring both dimensions to *the same generic unit.*

108. Hence it is obvious, that the product of any two integral numbers represents *the area of a parallelogram*—one number being its length and the other its breadth. Also, when the two numbers are alike, their product is *the area of a square,* and is, therefore, called *a square number.* Therefore any quantity multiplied into itself is *the square of that quantity.* Thus, $a \times a$ is the square of a, usually written $a\,a$ or a^2. The quantity itself, or length of one side, is called *the square root.* The small figure 2 on the right is called *the index,* and shows *how many times a is factor.*

109. The oblique line $a\,b$, which joins two opposite angles of the Table, is thence called *diagonal:* sometimes it is called *diameter,* because it metes or divides the square into two equal parts. This diameter passes through all the square numbers in the Table, which, taken from a to b, succeed each other, thus:

Squares -	1^2	4	9	16	25	36	49	64	81
Square Roots	1	2	3	4	5	6	7	8	9

At the top of the column, and on the left in the row in which any square number is found, we find its root, which is *the length of the side of the square,* the area of which the square number represents; that is, in each square, the square root or length of the side, is seen *in either of the opposite angles, through which the diameter does not pass.*

Note. The figure 1, as a root, is *the side of the square* or length of 1 inch. As a square it is *the area* or square inch; the generic unit by which the whole surface of the Table or that of any parallelogram or square in it is measured.

110. The unit, to which we compare, and by which we measure, any surface, is *always a square.* Thus the surface of the globe we inhabit, or of any country, kingdom, state, township, &c., is generally measured by *the square mile.* Most kinds of artificers' work, such as bricklaying, flooring, plastering, painting, glazing, &c. is measured by *the square foot or square yard.* Also the work, in all the branches, is *generally in the form of a parallelogram.* So that, in multiplying any two numbers, the scholar may suppose that he is finding *the content or area of a piece of artificer's work in the form of a parallelogram;* the product being *a number of square feet or square yards,* according to the *measure used in taking its length and breadth.* Suppose the above Table to represent a square township, each side of which is 9 miles: then, it is plain that the township contains 81 *square miles.* Again, suppose the Table a window, each side of which is 9 feet long; then, the glazing work contains 81 *square feet.* Again, suppose it a court-yard, the side of which is 9 yards long; then the paving work consists of 81 *square yards.*

111. Though the Pythagorean Table is sufficient for the multiplication of all numbers, an acquaintance with the following extension of it will, in many cases, be found advantageous. The student may, however, make use of it or not, as he pleases. We shall only remark that arithmeticians always multiply by 11 or 12, as by a single figure.

As the multiplication of a number by 10, consists in merely placing 0 on the right, this number, in the Table, is omitted as unnecessary.

To multiply by 11 any number expressed by two figures, the sum of which does not exceed 9, we have only to separate the figures and place their sum in the middle. Thus, to multiply 23 by 11, we separate the figures 2 and 3, and placing their sum 5 in the middle, we have 253 for the product. Also $32 \times 11 = 352$.

Extension of the Multiplication Table.

a

1	2	3	4	5	6	7	8	9	11	12	13	14	15	16	17	18	19	20
11	22	33	44	55	66	77	88	99	121	132	143	154	165	176	187	198	209	220
12	24	36	48	60	72	84	96	108	132	144	156	168	180	192	204	216	228	240
13	26	39	52	65	78	91	104	117	143	156	169	182	195	208	221	234	247	260
14	28	42	56	70	84	98	112	126	154	168	182	196	210	224	238	252	266	280
15	30	45	60	75	90	105	120	135	165	180	195	210	225	240	255	270	285	300
16	32	48	64	80	96	112	128	144	176	192	208	224	240	256	272	288	304	320
17	34	51	68	85	102	119	136	153	187	204	221	238	255	272	289	306	323	340
18	36	54	72	90	108	126	144	162	198	216	234	252	270	288	306	324	342	360
19	38	57	76	95	114	133	152	171	209	228	247	266	285	304	323	342	361	380
20	40	60	80	100	120	140	160	180	220	240	260	280	300	320	340	360	380	400

b

112. When a number subtracted from another number as often as possible leaves no remainder, it is called *a measure of that other number.* Also the number which contains another a certain number of times is called *its multiple.* Thus, 6 is a measure of 24, and 24 a multiple of 6.

A number which measures each of several numbers is called *their common measure.* Also a number which can be measured by each of several numbers is called *their common multiple.* Thus, 2 is *a common measure of all even numbers* 2, 4, 6, 8, 10, 12, &c. Also 24 is a common multiple of 2, 3, 4, 6, 8, and 12: hence, these numbers are all factors of 24. Now, if one of these be assumed as one of two factors of 24, *the other is arbitrary,* being the number which shows how many times the assumed number can be subtracted. Thus if we assume 12, the other factor is 2; if we assume 8, the other is 3.

113. To facilitate the acquisition of the above Table, as well as to acquaint the scholar with some useful properties of numbers, let us observe:

1. That any number multipled by 2 or any other even number is *an even number,* and, consequently, terminates with 2,

4, 6, 8 or 0. Hence, in the Table, the numbers in all the columns, except the last, under the even numbers, terminate with these figures, which succeed each other in some regular order.

2. That, as every alt unit is composed of *nines and a unit*, and, (as 9 is divisible by 3,) also, of *threes and a unit*, it is evident that an alt unit divided by 3 or by 9 will always leave *a unit* for remainder. Wherefore, each unit being considered a remainder, if the sum of the figures, in any number, be divisible by 3, *the number also is divisible by* 3: if the sum be divisible by 9, *the number is divisible by* 9; consequently, when any number is multiplied by 3 or by 9, *the sum of the figures of the product is a multiple of* 3 *or of* 9 *respectively*. Hence, the sum of the figures of any number in the column under 3, is a multiple of 3: under 9, the sum is 9 or its multiple. As we descend in this column, the unit's figure is diminished, and the tens figure increased by a unit at a time, (63.) Under 19, the unit figure diminishes in like manner, and the tens figure (on account of the ten on the left of 9) increases by 2 units at a time.

3. That, as 5 × 2 = 10, *five times any even number is a number of tens*, and terminates with 0. Also 5 *times any odd number must terminate with* 5; because, 5 times its even part is a number of tens, and 5 times the unit which renders it odd is 5. Hence the numbers under 5 and 15 terminate with 5 and 0 alternately.

4. That, under 11, the unit figure of each number agrees with the unit figure of the number multiplied by 11. That the tens figure is one unit greater, and that the left-hand figure is 1, except in the two lower numbers.

5. That, under 7 and 17, the unit figure, excepting 0, 1, and 2, diminishes by 3 units at a time, (because with the additional 7, this constitutes 10,) and under 13 increases by 3.

The oblique line *a b* passes through the square numbers, which, omitting 9, are

Squares -	121	144	169	196	225	256	289	324	361	400
Square Roots	11	12	13	14	15	16	17	18	19	20

114. When the multiplicand consists of several figures, and the multiplier of a single figure, we place the multiplier under the unit figure of the multiplicand; and, having drawn a line underneath, we multiply, successively, all the figures of the

multiplicand by the multiplier, beginning with the units. The unit figure of each product we write under the figure which gave it, and the tens we retain, as in addition, to add to the next product. Having multiplied the last or left-hand figure of the multiplicand, we write the whole product underneath, which completes the operation, the number under the line being the product of the two factors—that is, *the sum which would arise from the addition of either as often as the other contains a unit.*

1. To multiply 6342 by 6, we write the number thus:

$$\begin{array}{rl} 6342 & \text{multiplicand.} \\ 6 & \text{multiplier.} \\ \hline 38052 & \text{product.} \\ 221 & \end{array}$$

and, beginning with the units, we say: 6 times 2 is 12. Here, as in ordinary addition, we write 2 and carry 1: we then say, 6 times 4 is 24, and 1 is 25; we write 5 and carry 2: then, 6 times 3 is 18, and 2 is 20; we write 0 and carry 2: lastly, 6 times 6 is 36, and 2 is 38, the whole of which we write underneath, and have 38052 for the product.

If we write the number 6342 six times, thus,

$$\begin{array}{r} 6342 \\ 6342 \\ 6342 \\ 6342 \\ 6342 \\ 6342 \\ \hline 38052 \\ 221 \end{array}$$

and perform the addition, we have the same result.

Hence, it is evident that multiplication is *a species of addition;* consequently, we begin with the units, proceed towards the left, carry one for every ten, &c., as in ordinary addition, and for the same reasons.

Also, that if we add two or more unequal numbers, the operation is *an ordinary or common addition.* But if we add a number which is repeated any number of times, *the operation is properly a multiplication,* and in this consists the only essential difference in the nature of the operations.

115. If any number be multiplied by 0, *nought, the product*

will be nought; because it is evident that nought added to itself ever so many times can never produce any thing. Therefore, whenever, in multiplying, we meet with 0 in the multiplicand, and have nothing to carry from the product of the preceding figure, *we write* 0 *in the product.* Also, in multiplying the figure which precedes 0, if its product consist of two figures, we write it *as we should that of the last figure of the multiplicand.* If its product consist of three figures, *we carry only the left-hand figure, after we have written the two others underneath.*

2. $860142 \times 5 = 4300710$

$$\begin{array}{r} 860142 \\ 5 \\ \hline 4300710 \end{array}$$

Here, we say, 5 times 2 is 10; 0 and go 1; five times 4 is 20, and 1 is 21; 1 and go 2; 5 times 1 is 5, and 2 is 7; the next figure being 0, we write 0; then, 5 times 6 is 30; 0 and go 3; 5 times 8 is 40, and 3 is 43.

3. $6753 \times 2 = 13506$
4. $8492 \times 3 =$
5. $5467 \times 4 =$
6. $390835 \times 5 =$
7. $48706 \times 6 =$
8. $87965 \times 7 =$
9. $76009 \times 8 =$
10. $93584 \times 9 =$

11. Multiply 5384679 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, and find the sum of the several products.

Answer. Two hundred and thirty-six millions, nine hundred and twenty-five thousand eight hundred and seventy-six.

12. Multiply 93578864 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, and find the difference between the sum of the products and one trillion.

Answer. Nine hundred and ninety-five billions, eight hundred and eighty-two millions, five hundred and twenty-nine thousand nine hundred and eighty-four.

116. When the multiplier, as well as the multiplicand, consists of several figures, having placed the multiplier under the multiplicand and drawn a line underneath, first multiply all the figures of the multiplicand by the unit figure of the multiplier, as above. Then multiply by the tens in the same manner, placing the product under the first product, *so that its unit figure may stand under the tens of the first product.* Continue to multiply successively by the figures of the multiplier, always placing the first figure of each product *under that*

figure of the first product which stands in the same place of figures as the figure by which you multiply. Having multiplied by all the figures of the multiplier, add the several products, and the sum is the total product.

117. The reason why the products must be placed as above directed, is as follows: the multiplicand is multiplied by the several figures of the multiplier as though each of these expressed no more than simple units. But any figure towards the left is 10, 100, 1000, &c. times greater than if it expressed only simple units, according to its place; consequently, the product found in multiplying by it, as though it were simple units, should be made 10, 100, 1000, &c. times greater, which is done by removing this product one, two, three, &c. places towards the left. Hence, *the first figure of each product must stand as far towards the left as the figure which gave it.*

To find the product of 6574 × 432, we place the numbers thus:

6574	
432	
13148	twice the multiplicand.
19722	ten times 3 times, or 30 times.
26296	100 times 4 times, or 400 times.
2839968	product, 432 times 6574,

and, having multiplied by the unit figure 2, as usual, we multiply by the 3 tens in like manner, writing the first figure of this second product in the place of tens—that is, under the tens of the first product; lastly, we multiply by the 4 hundreds, writing the first figure of this product under the place of hundreds in the first product.

As any significant figure is ten times greater when removed one place towards the left, the second product, as it now stands, is ten times greater than if it had stood directly under the first; it is, therefore, 10 times 3 times, or 30 times the multiplicand. Again, the third product, which is 4 times, being removed two places, is 100 times as great as if it had stood directly under the first; it is, therefore, 100 times 4 times, or 400 times the multiplicand. Wherefore, as 400 + 30 + 2 = 432, the sum found by adding the three products just as they stand, is 432 times 6574, as was required.

118. When several numbers have the sign × *into,* between them, *we first multiply any two of them: we then multiply*

the product by another, and this product again by another, and so on, till all are involved.

1. 7 × 4 × 2 × 6 = 336, which is found by multiplying from left to right, thus: 7 × 4 = 28; then, 28 × 2 = 56; and, lastly, 56 × 6 = 336. Or, from right to left, thus: 6 × 2 = 12; 12 × 4 = 48, and 48 × 7 = 336, as before. *It is*, therefore, *of no consequence*, as to the result, in what order the numbers are multiplied. Also, either operation serves as proof to the other.

Let the numbers in the following examples be multiplied from right to left and from left to right. If the product be the same both ways, the work can scarcely be wrong.

2. 46 × 34 × 87 = 136068

$$\begin{array}{r} 46 \\ 34 \\ \hline 184 \\ 138 \\ \hline 1564 \\ 87 \\ \hline 10948 \\ 12512 \\ \hline 136068 \end{array} \qquad\qquad \begin{array}{r} 87 \\ 34 \\ \hline 348 \\ 261 \\ \hline 2958 \\ 46 \\ \hline 17748 \\ 11832 \\ \hline 136068 \end{array}$$

3. 26 × 53 × 47 × 9 =
4. 57 × 73 × 64 × 4 =
5. 384 × 58 × 97 × 7 =
6. 876 × 239 × 547 =
7. 4376 × 534 × 6892 =
8. 8394 × 2537 × 957 =

119. When there are ciphers between the figures of the multiplier, *we omit them and proceed as usual.*

1. 596843 × 50009 = 29847521587

$$\begin{array}{rl} 596843 & \\ 50009 & \\ \hline 5371587 & \text{nine times.} \\ 2984215 & \text{50000 times.} \\ \hline 29847521587 & \text{sum 50009 times.} \end{array}$$

Having multiplied by 9, we proceed at once, without regarding the ciphers, to multiply by 5, which is 50000, or 5 × 10000. We, therefore, place the unit figure of the product under the tens of thousands in the first product, by which means this product becomes 50000 times the multiplicand. Wherefore the sum of both products is 50009 times 596843.

Let the following be multiplied both ways, as in Art. 118 :

2. $908 \times 4805 \times 709 = 3093324460$
3. $59008 \times 2009 \times 4007 =$
4. $70063 \times 908 \times 70005 =$
5. $60007 \times 4072 \times 9905 =$
6. $87006 \times 800097 \times 580067 =$

120. If either or both of the factors terminate with ciphers, we cut off these ciphers and *multiply the numbers without them :* after which *we place as many ciphers on the right of the product as there were cut off from both factors.*

The reason for thus proceeding is obvious, if we consider that, for every cipher omitted on the right of either factor, that factor is (56) ten times less. Hence, the other factor is repeated a number of times which is ten times less than it should be; consequently, the product, which is ten times too small, must be multiplied by ten—that is, (55,) it must have the cipher which was omitted placed on the right.

1. $59400 \times 290 = 17226000$

$$\begin{array}{r} 594 \\ 29 \\ \hline 5346 \\ 1188 \\ \hline 17226 \end{array}$$

Having found the product of $594 \times 29 = 17226$, we reason thus :

$$\left.\begin{array}{rl} & 59400 = 594 \times 100 \\ \text{and} & 290 = 29 \times 10 \end{array}\right\} \quad (55)$$

consequently, $59400 \times 290 = 594 \times 100 \times 29 \times 10$, or $594 \times 29 \times 100 \times 10$, or $594 \times 29 \times 1000$; wherefore, the product 17226 must be multiplied by 1000; that is, (55,) we must place 3 ciphers on the right, which is the number of ciphers omitted on the right of the two factors. The true product, therefore, is 17226000.

Let the following multiplications be performed both ways :

2. $320 \times 600 \times 780 = 32 \times 6 \times 78 \times 10000 = 149760000$
3. $50900 \times 25000 \times 890 =$
4. $5903000 \times 5080 \times 6400 =$
5. $1976000 \times 400070 \times 70500 =$
6. $8900 \times 305007000 \times 30300 =$

121. A unit followed by ciphers is evidently less than any

other number containing the same number of figures, and (37) greater than any number containing one figure less. Now, if we multiply a number by a unit followed by ciphers, which is merely to place the ciphers following the unit on the right of the given number, the number of figures in the product will, in this case, be one less than the number in both factors. But the number of figures in the two factors can never exceed the number in the product by more than one figure; for, if it could, then might a number greater than a unit followed by ciphers give a less product, which is impossible.

Again, if we place as many ciphers on the right of the multiplicand as there are figures in the multiplier, we shall have just as many figures in the number thus formed, as there are in both factors. But this number is the product of the multiplicand multiplied (37) by a greater number than the multiplier, and is, therefore, greater than the true product. Wherefore, the product of two numbers cannot contain more figures than are contained in both factors; for, if it could, then might a less multiplier give a greater product, which is impossible.

From the above, therefore, we infer that *the product of two numbers contains as many figures as are contained in both, or one figure less.*

To predetermine whether there will be as many or one less, which is often useful, *multiply the left-hand figure of one factor by that of the other:* if the product exceeds 9, *there will be as many:* if, with what may be to carry from preceding figures, it cannot exceed 9, *there will be one less.*

122. We have seen (113) that when the sum of the figures of a number is a multiple of 3 or of 9, the number itself is a multiple of the same number. Now, it is evident that when either of two factors is a multiple of 3 or of 9, *their product will also be a multiple of the same number.* Wherefore, having found that one of the factors is a multiple of 3 or of 9, if we find that the product is not a multiple of the same number, this is a proof that *there is an error in the work which must be corrected.* If we find that the product is also a multiple, though this is not absolute, it is presumptive evidence that the work is right. The fallible cases are *when the error is the number by which we are proving, or its multiple;* or, *when errors balance each other,* neither of which cases is very probable.

123. If we multiply the binomial value of an alt unit (

by any figure less than 9, we shall have the binomial value of that figure as *an alt figure of the same order as the alt unit.* But, in multiplying the nines of the alt unit, we still have a multiple of 9, and in multiplying the unit, we have the figure by which we multiply, considered as single units. Wherefore, if all the nines be taken out of the value of an alt figure less than 9, *the remainder will always be the figure itself.* If, then, we add all the figures of a number together, and take the nines out of the sum, we shall have *the remainder that would result in subtracting* 9 *from the number as often as possible.* This is called casting out the nines; in peforming which *we omit* 9 *or the sum of any two figures which make* 9.

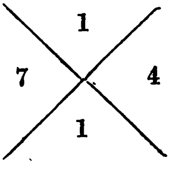
To cast the nines out of 7468935, we omit 9, also the figures 6 and 3, 5 and 4; we then say, 7 and 8 is 15, from which we cast 9, by saying, 1 and 5 is 6. Hence, 6 is the remainder that would result in subtracting 9 as often as possible from 7468935, which is *the same as to divide by* 9.

Proof. 7468935—6=7468929, which is a multiple of 9.

124. When neither factor is a multiple of 3 or 9. To prove the work of a multiplication, we cast the nines out of both factors, and place the two remainders in the opposite angles of a cross in the form of the sign *into.* We then multiply the two remainders as the sign indicates, cast the nines out of their product, and place the remainder in the upper angle. Lastly, we cast the nines out of the total product, and place the remainder in the lower angle. If the number in the lower angle agrees with that in the upper, the work is supposed to be right.

$$\begin{array}{r} 6988543 \\ 58 \\ \hline 55908344 \\ 34942715 \\ \hline 405335494 \end{array}$$

Proof.



We first cast the nines out of 6988543, and 7 remains: we then cast the nines out of 58, and 4 remains. These two remainders, 7 and 4, we place in the side angles of the cross. We then multiply them together, and cast the nines out of their product 28, saying: 2 and 8 is 10; 1 and 0 is 1, which we place in the upper angle. Lastly, we cast the nines out of the product 405335494, and as the remainder 1 agrees with the 1 in the upper angle, we suppose the work to be right.

Let $x =$ all the nines contained in 6988543; that is, let $x = 6988536$, which is a multiple of 9. Then the binomial value of 6988543 is $x + 7$. Again: let $y = 54$, the number consisting of all the nines in 58. Then the bimonial value of 58 is $y + 4$. Now if, instead of the factors, we multiply their binomial values, we shall have an expression consisting of 4 parts, which show how the true product 405335494 could be formed by substituting, for x and y, their numerical values.

$$\begin{array}{r} x + 7 \\ y + 4 \\ \hline xy + 7y + 4x\,(4 \times 7) \end{array}$$

In this product xy represents the nines of the two factors multiplied together: $7y$, the nines of the multiplier multiplied by 7: $4x$, the nines of the multiplicand multiplied by 4; and, lastly, 4×7, the product of the two parts not composed of nines. That is, $xy + 7y + 4x + (4 \times 7) = 6988536 \times 54 + 7 \times 54 + 4 \times 6988536 + (4 \times 7) = 405335494$. Wherefore, as $xy + 7y + 4x$ is a number of nines, the only part of the product 405335494, which is not composed of nines, is the product of 4×7; wherefore, the remainder found in casting the nines out of the product will always be the same as the remainder found in casting the nines out of the product of the numbers placed in the side angles of the cross. This proof, as has been shown, (122,) is not infallible, but, as the cases of its failure are extremely rare, it is practised by all mathematicians, and considered a valuable verification.

Convenient Contractions.

125. To multiply a number by 9, or by any number of nines, multiply the given number by the alt unit of the multiplier, and from the product subtract the given number; that is, *place as many ciphers on the right of the given number as there are nines to multiply by, and from the number thus formed subtract the given number.* The alt unit being just one unit greater than the multiplier, it is evident that in placing the ciphers on the right, we repeat the number once

too often; therefore, in subtracting it, we have [illegible] [illegible] result.

1. $7854 \times 9 = 78540 - 7854 = 70$[illegible]

78540 ten times.
7854 once.

70686 the difference is 9 times.

But, as we here observe that the first figure is under 0, and each of the other figures under its right-hand figure; also that the left-hand figure is projected one place towards the left, we may place the numbers as usual thus:

$$\begin{array}{r} 7854 \\ 9 \\ \hline 70686 \end{array}$$

and say, 4 from 10, *six:* then, subtracting each figure from the figure on the right, and taking care, when there is one to carry, to read the left-hand figure one greater, we say, 6 from 14, *eight;* 9 from 15, *six;* 8 from 8, *nought;* and, lastly, as the left-hand figure is projected one place towards the left, we write it down, saying, *seven.* When there is one to carry, *we subtract it from the left-hand figure, and write the remainder.* Prove, as directed Art. 122.

2. $412587 \times 9 = 3713283$

$$\begin{array}{r} 412587 \\ 9 \\ \hline 3713283 \end{array}$$

Prove the work as above.

Here we say, 7 from 10, *three;* 9 from 17, *eight;* 6 from 8, *two;* 2 from 5, *three;* 1 from 2, *one;* 4 from 11, *seven;* 1 from 4, *three.*

3. $7854 \times 99 = 785400 - 7854 = 777546$

$$\begin{array}{r} 7854 \\ 99 \\ \hline 777546 \end{array}$$

Here we have two ciphers on the right, and, consequently, two figures projected towards the left: therefore, under 54, we write its ar. comp. 46, and, as there is always 1 to carry, which is the alt of the multiplier, we add 1 to 8, and say, 9 from 14, *five;* then, adding 1 to 7, 8 from 15, *seven.* Now, if there were no unit to carry, we should write 78, the number projected; but, as there is, we say, 1 from 8, *seven;* and, lastly, we write 7 on the left, which completes the operation.

4. 3141592 × 99999 = 314156058408
3141592
99999
314156058408. Prove by Art. 122.

Having written the ar. comp. of 41592, which is 58408, we say: 2 from 2, *nought;* 3 from 9, *six:* then, writing 31415, the part projected, on the left, the work is complete.

5. 2391786 × 9 =
6. 21526074 × 9 =
7. 193734666 × 9 =
8. 1743611994 × 9 =
9. 15692507946 × 9 =
10. 5271 × 99 =
11. 521829 × 999 =
12. 521307171 × 999999 =

126. To multiply a number by 11, 111, 1111, &c.

1. 7854 × 11 = 86394
78540 ten times.
7854 once.
86394 sum, or 11 times 7854.

Here, we see that the first figure must always be written as it is, and that in every other respect we proceed as in multiplying by 9, with this difference, instead of subtracting, we add. We may, therefore, place the numbers, as usual:

7854
11
86394

and say, 4, writing it as it is; then, 5 and 4 is 9; 8 and 5 is 13; 8 and 8 is 16; 1 and 7 is 8. With a little practice this is done as quickly as the figures can be written, and with scarcely any mental effort. Thus:

2. 5946258317
11
65408841487

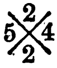
we say, in rapid succession seven, eight, four, eleven, fourteen, eight, eight, ten, fourteen, fifteen, six.

Here, for 11, we have added continually each figure to the next figure on the right; that is to say, *two consecutive figures.*

For 111, *we should add three;* for 1111, *four;* and, in general, *as many as there are figures in the multiplier.*

Observe, that having added the left-hand figure to the required number of figures on the right, *we drop one figure on the right,* and add the left-hand figure again to the remaining ones on the left of the figure we have dropped; we then *drop another, and add again, and so on, till we have the left-hand figure only, which we write on the left, as usual.*

3. 4592768 × 1111 = 5102565248



As the scholar already knows the form, we shall, in adding, mention only the results. Having written 8 on the right, we say, 6, 14; 8, 14, 22; 4, 11, 17, 25; 11, 13, 20, 26; 7, 16, 18, 25; 6, 11, 20, 22; 6, 11, 20; 6, 11; and, lastly, 5.

The scholar may prove this and the succeeding examples by multiplying in the ordinary way, and also by casting out the nines:

4. 162 × 11 × 11 × 11 × 11 × 11 = 26090262
5. 9261 × 11 × 11 × 11 × 11 × 11 = 1491493311
6. 5236 × 111 =
7. 581196 × 111 =
8. 64512756 × 111 =
9. 7160915916 × 1111 =

127. By a method analogous to that given for multiplying by 11, we multiply by any of the numbers 13, 14, 15, 16, 17, 18, 19. For example, to multiply 2568 by 14, we place the numbers thus:

$$\begin{array}{r} 2568 \\ 14 \\ \hline 35952 \end{array}$$

and beginning with the units, we say, 4 times 8 is 32; 2 and go 3; then, 4 times 6 is 24 and 3 is 27: to this we also add the figure 8 on the right, which makes 35; 5 and go 3: we then say, 4 times 5 is 20, and 3 is 23, and 6 (always adding the right-hand figure) is 29; 9 and go 2; then, 4 times 2 is 8, and 2 is 10, and 5 is 15; 5 and go 1, which, as there are no more figures, we add to the last figure 2, and placing the sum 3 on the left, we have 35952, the required product.

The reason for operating thus is the same as that given for the multiplication by 11. For, as we there add once to ten times, we here, in like manner, add four times to ten times. The scholar will clearly see the nature of the operation by multiplying in the usual way, which will also serve as proof.

1. $3141 \times 13 \times 14 \times 15 \times 16 = 137198880$
2. $5236 \times 16 \times 17 \times 18 \times 19 = 487073664$

128. To multiply by any of the numbers 21, 31, 41, 51, 61, 71, 81, 91, *write the unit figure of the multiplicand in the product; then multiply each figure of the multiplicand by the tens of the multiplier, beginning with the units, and to the product of each figure (besides what is carried) add the next figure of the multiplicand on the left.* Having multiplied the last figure, *set down the amount.*

. $7854 \times 81 = 636174$

$$\begin{array}{r} 7854 \\ 81 \\ \hline 636174 \end{array}$$

We first say, 4, writing it in the product. Then, 8 times 4 is 32, and 5 on the left is 37; 7 and go 3; 8 times 5 is 40, and 3 is 43, and 8 is 51; 1 and go 5; 8 times 8 is 64, and 5 is 69, and 7 is 76; 6 and go 7; 8 times 7 is 56, and 7 is 63: the whole of which we write on the left.

We shall see the reason for the above method by performing the operation in the usual way, thus:

$$\begin{array}{r} 7854 \\ 81 \\ \hline 7854 \\ 62832 \\ \hline 636174 \end{array}$$

and observing that, in all such cases, the unit figure of the multiplicand must be the unit figure of the product; also, that the unit figure of the second product is added to the tens of the first, (that is to say, to the tens of the multiplicand,) the tens to the hundreds, &c.

2. Multiply 293047586 by each of the numbers 9, 11, 13, 14, 15, 16, 17, 18, 19, by the methods given above, and find the difference between the sum of the products and one trillion, nine hundred and one millions, thirty thousand and six.

Ans. Nine hundred and sixty-two billions, two hundred and eighteen millions, seven hundred and forty-eight thousand six hundred and fifty-four.

3. Multiply 71930082546 by each of the numbers 21, 31, 41, 51, 61, 71, 81, 91, and find the difference between the sum of the products and one hundred quadrillions, nine trillions, twelve billions, one hundred and forty thousand, three hundred and sixty.

Ans. Ninety-nine quadrillions, nine hundred and seventy-six trillions, seven hundred and eighty-seven billions, three hundred and twenty-three millions, one hundred and fifty-nine thousand, seven hundred and fifty-two.

4. Multiply 670092099 by each of the numbers 9, 99, 999, and find the sum of the products.

Ans. Seven hundred and forty-one billions, seven hundred and ninety-one millions, nine hundred and fifty-three thousand, five hundred and ninety-three.

129. When the multiplier is the product of two numbers, neither of which is too great to multiply by at once, we may, if we please, *first multiply by one of them, and then multiply the product by the other.*

1. $24958 \times 132 = 24958 \times 12 \times 11 = 3294456$

$$\begin{array}{r l} 24958 & \\ 12 & \\ \hline 299496 & \\ 11 & \\ \hline 3294456 & \text{11 times 12 times, or 132 times.} \end{array}$$

In the same manner perform the following examples:

2. $392877 \times 25 =$
3. $864396 \times 49 =$
4. $2988657 \times 64 =$
5. $367948 \times 56 =$
6. $9436599 \times 72 =$
7. $84951645 \times 144 =$

8. Multiply 239548167 by each of the numbers 27, 28, 36, 45, 72, 84, and find the ar. comp. of the sum of the products.

Ans. Thirty billions, fifty-one millions, nine hundred and thirty-five thousand, two hundred and thirty-six.

130. By a method similar to that given Art. 127, we multiply at once by any of the numbers 112, 113, 114, 115, 116, 117, 118, 119. We have only to observe that instead of adding, besides what we carry, simply the right-hand ıre, we must, in this case, add *the two nearest figures on right.*

1. $63792 \times 118 = 7527456$

$$\begin{array}{r}63792\\118\\\hline 7527456\end{array}$$

Beginning with the units, we say, 8 times 2 is 16; *six* and go 1; 8 times 9 is 72, and 1 is 73, and 2 is 75; *five* and go 7; 8 times 7 is 56, and 7 is 63, and 9 is 72, and 2 is 74; *four* and go 7; 8 times 3 is 24, and 7 is 31, and 7 is 38, and 9 is 47; *seven* and go 4; 8 times 6 is 48, and 4 is 52, and 10 $(3+7)$ is 62; *two* and go 6 to 6 is 12, and 3 is 15; *five* and go 1 to 6 is 7.

By performing the operation in the usual way, the scholar will easily perceive that we have here taken the multiplicand 8 times plus 10 times plus 100 times, as was required.

Let the following multiplications be performed in the same manner:

2. $243 \times 111 \times 112 \times 113 \times 114 = 38916212832$
3. $612 \times 115 \times 116 \times 117 \times 118 \times 119 = 13412881329120$

131. Again, by a method similar to that given Art. 128, we multiply at once by any of the numbers 211, 311, 411, 511; 611, 711, 811, 911. For example, to multiply 59463 by 711, we write the numbers as usual,

$$\begin{array}{r}59463\\711\\\hline 42278193\end{array}$$

and say, 3 is *three;* 3 and 6 is *nine;* then, 7 times 3 is 21, and 6 is 27, and 4 is 31; *one* and go 3; 7 times 6 is 42, and 3 is 45, and 4 is 49, and 9 is 58; *eight* and go 5; 7 times 4 is 28, and 5 is 33, and 9 is 42, and 5 is 47; *seven* and go 4; 7 times 9 is 63, and 4 is 67, and 5 is 72; *two* and go 7; 7 times 5 is 35, and 7 is 42, which we write entire.

The scholar may prove the work by multiplying in the ordinary way, which will sufficiently illustrate the above method.

2. $513 \times 211 \times 311 \times 411 \times 511 = 7070057265033$
3. $234 \times 611 \times 711 \times 811 \times 911 = 75104489687994$

132. Also by a combination of the methods given in Arts. 127 and 128, we multiply, in one line, by any of the numbers

121, 131, 141, 151, 161, 171, 181, 191. For example, to multiply 478152 by 151, after placing the numbers

$$\begin{array}{r} 478152 \\ 151 \\ \hline 72200952 \end{array}$$

we say, 2 is *two;* 5 times is 2 is 10, and 5 is 15; *five* and go 1; 5 times 5 is 25, and 1 is 26, and 2 is 28, and 1 is 29; *nine* and go 2; 5 times 1 is 5, and 2 is 7, and 5 is 12, and 8 is 20; *nought* and go 2; 5 times 8 is 40, and 2 is 42, and 1 is 43, and 7 is 50; *nought* and go 5; 5 times 7 is 35, and 5 is 40, and 8 is 48, and 4 is 52; *two* and go 5; 5 times 4 is 20, and 5 is 25, and 7 is 32; *two* and go 3 to 4 is *seven.*

A close examination of Articles 127 and 128, as well as the proof in multiplying by the ordinary method, will be sufficient elucidation.

2. 315 × 121 × 131 × 141 × 151 = 106307346915
3. 621 × 161 × 171 × 181 × 191 = 591051778821

133. When one part of the multiplier is a multiple of another part of it, the work may often be very much contracted by, first, multiplying by the lesser part, and then multiplying the product thus obtained by the number which shows how many times this lesser part is contained in its multiple. This last is *the product of the multiplicand multiplied by the multiple.*

1. 66827394×13617=909988624098

$$\begin{array}{rl} 66827394 & \\ 13617 & \\ \hline 1136065698 & \text{17 times the multiplicand.} \\ 9088525584 & 8\times17\times100\text{, or 13600 times.} \\ \hline 909988624098 & \end{array}$$

and having multiplied by 17, we have 1136065698 for the first partial product. Then, as the part 136, of the multiplier is a multiple of 17, seeing that it contains 17 just eight times, we multiply the first product 1136065698 by 8, and have 9088525584 for 8 times 17 times, or 136 times the multiplicand. The units figure of this last product we place under the hundreds of the first, which makes the last product signify 13600 times the multiplicand. The sum of the two products is therefore 13617 times, as was required.

2. 358974×1664 = 597332736

$$\begin{array}{rl} 358974 & \\ 1664 & \\ \hline 5743584 & \text{sixteen hundred times the multiplicand.} \\ 22974336 & \text{sixty-four times.} \\ \hline 597332736 & \text{product, or 1664 times.} \end{array}$$

As 16, the part of the multiplier on the left, is a measure of 64, the remaining part, we first multiply by 16, and by removing the product two places towards the left, so that its unit figure may stand in the place of hundreds, we have 1600 times the multiplicand. Then, as the part 64 is a number of single units, we multiply 5743584, which is 16 times the multiplicand, by 4, and by placing the unit figure of this product two places farther to the right, we have simply 64 times the multiplicand. The sum of the two products is therefore 1664 times the multiplicand, as was required.

Let the value of the following expressions be found according to the above methods of contraction; also, let them, by way of proof, be multiplied *from left to right and from right to left:*

3. 243×9×11×13=
4. 4214×99×486=
5. 1272×9999×1836=
6. 1451×811×3913=
7. 242121×7813×13211=
8. 181111×12999×4515=
9. 11214×99999×1751=
10. 6111222×19118×16814=

134. A vinculum is *a bar* ———, *or parenthesis* (), used *to connect several numbers or quantities, so that the whole may, as one quantity, be subjected to any of the cardinal operations.* Thus $\overline{8+4}\times 2$ or $(8+4)2$, signifies that the sum of 8 and 4 is to be multiplied by 2. Now 12, the sum of 8 and 4, multiplied by 2 is (59 and 101) $12+12$, and this is evidently equal to $(8+4)+(8+4)$ or $8+8+4+4$: that is, to twice 8 plus twice 4; wherefore $(8+4)2=16+8$. Now if we resolve the number 12 into any other parts as $7+3+2$: then $(7+3+2)3$, is evidently (101) equal to $(7+3+2)+(7+3+2)+(7+3+2)$ or $7+7+7+3+3+3+2+2+2$; that is, to 3 times 7, plus 3 times 3, plus 3 times 2: wherefore $(7+3+2)3=21+9+6$. Hence we infer that *when a number is multiplied by any other number, the product is equal to the sum of the products when all its parts are separately multiplied by that other number,* what-

ever may be the parts or the multiplier. Or, as the sum of several numbers is a number whereof the numbers are parts, we say, that *when the sum of several numbers is multiplied by any number, the product is equal to the sum of the products when each number is separately multiplied by that number, and conversely.* Wherefore $9.3 + 6.3 + 5.3 + 2.3 = (9 + 6 + 5 + 2)3$. The sign *into*, made thus [.], though not so commonly used as the other, is, in many cases, very commodious.

Also $(8 - 4)3$ signifies that the difference between 8 and 4, or 4, is to be multiplied by 3. Now $4 . 3$ or $4 + 4 + 4 = (8 - 4) + (8 - 4) + (8 - 4)$, or $8 + 8 + 8 - 4 - 4 - 4$ or 3 times 8 minus 3 times 4: wherefore $(8 - 4)3 = 24 - 12$. Hence we infer that when the difference between two numbers is multiplied by any number, *the product is equal to the difference of the products when each number is separately multiplied by that number:* consequently, if there be several positive and several negative numbers under the vinculum to be multiplied: as the sum of the positive products is one number and the sum of the negative another, it is plain that the above elucidation will apply to them in their combined form, and therefore the value of the expression will equal the several products found in multiplying each number by the multiplier and writing the products in succession, each with its appropriate sign. Thus, $(9 + 8 - 5 - 6)5 = 45 + 40 - 25 - 30$. When the vinculum is thus removed by writing the several products, *the expression is said to be expanded.*

To reduce such an expression, or, in other words, *to find its value,* proceed thus:

$(9 + 8 - 5 - 6)5 = 45 + 40 - 25 - 30 = 85 - 55 = 30$, the value of the expression.

135. Again $8 - \overline{4 + 2}$, or $8 - (4 + 2)$, signifies that the sum of 4 and 2 is to be taken from 8. Also $8 - \overline{4 - 2}$ or $8 - (4 - 2)$, signifies that the difference between 4 and 2 is to be subtracted from 8. Now if from the expression $8 - (4 + 2)$, we remove the vinculum, we have $8 - 4 + 2$; but this last does not mean that the sum of 4 and 2 is to be taken from 8, but that 4 only is to be subtracted from the sum of 8 and 2, and thus the number 2, instead of a number to be subtracted, becomes by its sign, a number to be added. Wherefore, that the expression $8 - (4 + 2)$ may retain its value,

we must, in removing the vinculum, change the sign of the number 2 *from plus to minus,* thus $8 - \overline{4 + 2} = 8 - 4 - 2$.

If from the expresssion $8 - (4 - 2)$, we remove the vinculum, we have $8 - 4 - 2$; but this last does not mean that the difference between 4 and 2 is to be taken from 8, but their sum; therefore, as in subtracting 4 we take away 2 too much, we must add 2 to make up the deficiency; consequently, we must, in removing the vinculum, change the sign of the number 2 *from minus to plus:* thus $8 - \overline{4 - 2} = 8 - 4 + 2$.

Hence we see that in removing a vinculum preceded by minus, *the sign of a positive number under it must be changed from plus to minus, and that of a negative one from minus to plus.* Observe, *that the sign preceding the vinculum must not be changed.*

136. When there are several positive and several negative numbers under a vinculum *preceded by minus,* as the sum of all the positive numbers is one positive number, and the sum of all the negative, one negative number, the above reasoning applies to the whole of them; consequently, in removing the vinculum, *the signs of all the numbers under it must be changed, the positive to negative and the negative to positive, the sign preceding the vinculum remaining the same.* Thus:

$$967 - (19 + 7 - 8 + 3 - 11)\,7 = 967 - 133 - 49 + 56 - 21 + 77 = 1100 - 203 = 897.$$

The student must carefully observe, that when we would remove a vinculum preceded by plus, (98,) *the signs of the numbers under it must not be changed.* For example:

$$967 + (19 + 7 - 8 + 3 - 11)\,7 = 967 + 133 + 49 - 56 + 21 - 77 = 1170 - 133 = 1037.$$

137. Any two numbers having the sign *into* between them, are considered *as having immediate connection with each other, unless this is otherwise determined by the vinculum.* Thus:

$6 + 4 \times 3 = 18$ and $4 + 6 . 3 = 22$; but $\overline{6 + 4} \times 3 = 30$, and $(4 + 6)\,3 = 30$. Again $6 - 2 \times 3 = 0$, but $(6 - 2)\,3 = 12$.

138. From what has been said the student will perceive that when two numbers are to be multiplied, the separation of either or both into parts, at pleasure, has *no effect* upon the product. For example, $17 \times 13 = 221$. Now we may, if

we please, separate 17 into the parts 8, 6 and 3; and 13, into the parts 7, 4 and 2, and proceed thus:

$$\begin{array}{ll}
\quad 8+6+3 & \\
\quad \underline{7+4+2} & \\
56+42+21 & \text{product by } 7 \\
\qquad\quad 32+24+12 & \qquad\qquad 4 \\
\qquad\qquad\qquad \underline{16+12+6} & \qquad\qquad 2 \\
\hline
56+42+21+32+24+12+16+12+6 & \text{product by } 13
\end{array}$$

The sum of all these products is 221, as before.

Let the following expressions be reduced according to the examples given above:

1. $32-\overline{17+6-14-9}=32$
2. $186+43-27-\overline{6+8+4}=184$
3. $9645-(1736+23)-17\times 6=778$
4. $9645-1736+(23-17)6=7945$
5. $9645-(1736+23-17)6=-807$
6. $\overline{9645-1736+23-17}\times 6=47490$

SECTION VI.

DIVISION.

139. DIVISION is the operation by which we find *how many times a less number is contained in a greater more easily than by several subtractions.* Thus, if we would divide 24 by 6, that is, if we would find how often 6 is contained in 24, instead of operating by several subtractions (60), we recollect (63) how many sixes must be added together to make 24, and say at once 6 is contained 4 times: therefore 4 is the answer.

Also, when a number is given to be divided into a certain number of equal parts, the value of each part is found by division. For if, in the above example, it be required to divide the number 24 into 6 equal parts, we recollect (63) that $6+6+6+6=24$; that is to say, that 4 times 6 is 24; and again (59) that this is the same as 6 times 4; that is, $4+4+4+4+4+4=24$: consequently, the number 24 is divided into 6 equal parts, the value of each part being 4.

140. *The number from which we subtract,* and which is consequently divided or separated into equal parts, is called *the dividend.*

The number which we repeatedly subtract, and which naturally signifies the value of each of the equal parts into which the dividend is separated, is called *the divisor*.

The number which shows *how many times the divisor can be subtracted*, or *how often it is contained in the dividend*, and hence *the number of equal parts into which, by the operation, the dividend is separated*, is called *the quotient.*

Let us here observe that, *where the division is exact, either the divisor or quotient may signify the number of equal parts into which the dividend is separated;* the remaining one being the value of each part. Thus, if the divisor signifies the number of parts, the quotient is the value of each part; and, if the divisor signifies the value of each part, the quotient is the number of parts.

141. From what has been said (60 and 139) it is plain that, as *the dividend is the product of the divisor and quotient*, a division may always be proved *by multiplying the divisor and quotient together, which will give the dividend* when the work is right.

NOTE. Though this proof is infallible, whatever may be the quotient, the scholar is here only required to make use of it when the divisor is contained an exact number of times.

142. The following Table will tend to enable the scholar to divide numbers with facility. Let each division in the table (where the divisor and quotient are not alike, that is, where the dividend is not the square of the divisor) be performed both ways: that is, having found the quotient, *divide by the quotient, which will give the divisor*. Thus, for the expression $6 \div 2 = 3$, say 2 in 6 three times; 3 in 6 twice: for the next $8 \div 2 = 4$, say 2 in 8 four times; 4 in 8 twice, and so on throughout the table. Hence, the student may infer that he can always prove an exact division *by another division.*

DIVISION TABLE.

$4 \div 2 = 2$	$9 \div 3 = 3$	$16 \div 4 = 4$	$25 \div 5 = 5$
$6 \div 2 = 3$	$12 \div 3 = 4$	$20 \div 4 = 5$	$30 \div 5 = 6$
$8 \div 2 = 4$	$15 \div 3 = 5$	$24 \div 4 = 6$	$35 \div 5 = 7$
$10 \div 2 = 5$	$18 \div 3 = 6$	$28 \div 4 = 7$	$40 \div 5 = 8$
$12 \div 2 = 6$	$21 \div 3 = 7$	$32 \div 4 = 8$	$45 \div 5 = 9$
$14 \div 2 = 7$	$24 \div 3 = 8$	$36 \div 4 = 9$	$36 \div 6 = 6$
$16 \div 2 = 8$	$27 \div 3 = 9$	$49 \div 7 = 7$	$42 \div 6 = 7$
$18 \div 2 = 9$	$64 \div 8 = 8$	$56 \div 7 = 8$	$48 \div 6 = 8$
$81 \div 9 = 9$	$72 \div 8 = 9$	$63 \div 7 = 9$	$54 \div 6 = 9$

143. If a unit or whole thing be divided into *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, &c. equal parts, the parts are called *halves*, *thirds*, *fourths*, *fifths*, *sixths*, *sevenths*, *eighths*, &c. In each successive division one part is called *one-half*, *one-third*, *one-fourth*, *one-fifth*, *one-sixth*, *one-seventh*, *one-eighth*, &c. These parts are expressed in figures thus $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, &c.

144. The divisor being the number of equal parts and the quotient the value of each; if we divide a number by 2, the quotient is *one-half* that number. If we divide by 3, the quotient is *one-third;* and, in general, to divide a number by 4, 5, 6, 7, 8, &c., is to take $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, &c. of that number. Also inversely, to take *one-half*, *one-third*, *one-fourth*, &c. of any number, is to divide that number by 2, 3, 4, &c. The expressions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. signify 1 *divided by* 2; 1 *divided by* 3, &c.

145. As any number or quantity contains itself once, it is plain that a unit divided by a unit is still a unit; also, that a unit is *the value of any quantity divided by itself*: thus, $\frac{2}{2}=1$; $\frac{3}{3}=1$; $\frac{a}{a}=1$, &c. As the lower number is always the divisor, it is plain that when this is greater than the upper one, the quotient is *less than a unit.* In this case the quantity is called *a fraction.* Thus $\frac{4}{5}$, which is read *four-fifths*, 4 divided by 5, or the fifth part of 4, is *a fraction.* The two numbers which express a fraction are called its *terms.*

146. As a single unit may be divided or separated into any number of equal parts, at pleasure; any two numbers may be assumed as dividend and divisor; consequently, *the quotient*, so far from being always an exact or whole number, *will more frequently be fractional*, and sometimes *merely a fraction.* For example, if we would find how often 9 is contained in 58, we easily perceive (123) that 58 is not an exact multiple, and that the remainder will be 4. Also, as the alt 5 is (123) 5 times 9, plus 5, and, as the 8 together with this 5 exceeds 9 by 4, it is evident that 58 contains 9 six times and 4 over. We express the division of this remainder by 9 thus, $\frac{4}{9}$. Wherefore, the operation is read 58 divided by 9 equals 6 and 4 ninths; and written thus, $\frac{58}{9}=6\frac{4}{9}$.

147. From the preceding we infer that, to find how often 9 is contained in a number expressed by two figures, the sum of which equals or exceeds 9, we have only to *add a unit to the left-hand figure.* Also that *the left-hand figure itself* is

the number of times, when the sum of the two is less than 9. Thus $\frac{73}{9} = 8\frac{1}{9}$; and $\frac{32}{9} = 3\frac{5}{9}$.

148. For more examples, the student may take the numbers that come between those of the division table. Thus, instead of saying 2 in 4 twice, he may say 2 in 5 twice and 1 over; 2 in 7 three times and 1 over. At the next column, instead of saying 3 in 9 three times, he may say 3 in 10 three times and 1 over; 3 in 11 three times and 2 over, and so on for the others. Again, he may say, $\frac{5}{2} = 2\frac{1}{2}$, five by 2 equals two and a half: $\frac{7}{2} = 3\frac{1}{2}$, &c.; $\frac{10}{3} = 3\frac{1}{3}$; $\frac{11}{3} = 3\frac{2}{3}$; $\frac{13}{3} = 4\frac{1}{3}$; $\frac{16}{3} = 5\frac{1}{3}$; $\frac{17}{3} = 5\frac{2}{3}$, &c. &c. This will be a valuable exercise.

149. Division presupposes *the subtraction of the divisor from the dividend as often as possible:* but this subtraction is possible as long as there remains a part of the dividend which equals or exceeds the divisor; and impossible, as soon as this remainder is less than the divisor, even by one unit. Wherefore, the ultimate remainder, when there is one, must be less than the divisor, and may be *any number in the natural scale,* 1, 2, 3, 4, 5, &c., *including the number which is less than the divisor by only one unit.* Thus, in dividing by 2, the remainder, if any, is always 1. In dividing by 3, it may be 1 or 2. In dividing by 4, it may be 1, 2 or 3, and, in like manner, for any other divisor.

150. When a number is given to be separated into a certain number of equal parts, the greater the given number is, the greater will be the value of each part; hence, if we increase the dividend any number of times, leaving the divisor the same, the quotient will be increased the same number of times; that is, (leaving the divisor the same,) *to multiply the dividend is to multiply the quotient.* Thus, as 6 contains 2 three times, 6 + 6 contains 2 three times plus three times: that is, twice 6 contains 2 twice as often as 6 contains 2, and it is plain that whatever number of times 6 is repeated, the quotient 3 will be repeated the same number of times. Hence, if we divide 6 tens by 2 the quotient will be 3 tens. If we divide 6 hundred by 2, the quotient will be 3 hundred, &c. Now it is evident, that the same reasoning will apply to any two numbers; wherefore, in general, *the quotient figure is always of the same order or name as the part divided,* which order (48) is determined *by that of the right-hand figure of the part divided.*

151. As no quotient figure of a superior order can arise

from the division of an inferior, and as the remainder resulting from the division of any superior order is (48) easily resolved into a number of units of any of the inferior orders, we always begin division *at the left hand.*

To divide a number consisting of several figures by any number not exceeding 12.

1. Take the smallest number of the left-hand figures of the dividend that will contain the divisor: that is, take as many as are contained in the divisor, or, when this is not sufficient, (121,) only *one figure more.*

2. Seek how often the divisor is contained in this part.

3. Place the number of times, which cannot exceed 9, in the quotient.

4. Multiply the divisor by the quotient figure.

5. Subtract the product, thus found, from the part divided.

6. To the right of the remainder annex the next figure of the principal dividend towards the right, and proceed to take all the steps, as before.

Having taken a number of the left-hand figures of the dividend sufficient to contain the divisor, we place the number of times that the divisor is contained in this part, in the quotient, under the unit figure of the part taken. Now this quotient figure, being (150) *of the same order as the unit figure of the part divided,* must have *as many figures on the right of it as there are figures remaining in the dividend.* For this reason, when any of the partial dividends is too small to contain the divisor, *we must place a cipher in the quotient,* annex one more figure to the partial dividend, (if any remain,) and proceed as before.

152. As the federal coin of the United States is constructed upon the same principle as ordinary numbers, ten units of each lower order being equal to one unit of the next higher, (see Table of Federal Money,) we shall take an example in this coin to illustrate the division of those numbers.

Suppose we have 71436 mills to divide equally among 6 persons, we distinguish the different orders of Federal Money in this number, thus: the lowest order of units, expressed by the figure 6 is mills: each unit of the next order, on the left, being ten of those, is one cent; the figure 3 is, therefore, 3 cents. Again, each unit in the 4, being ten cents, is one dime: the 4 is, therefore, 4 dimes. The 1, being ten dimes, is one dollar. Lastly, as each unit in the 7 is ten dollars, or one eagle, the 7 is 7 eagles. To perform the operation, we place the divisor 6 on the left of the dividend and draw a line underneath, thus:

Divisor 6) 71436 dividend.
11906 quotient.

Then, beginning at the left, as 6 is contained in 7 *once*, we write 1 in the quotient, under 7. This 1 shows that each person will receive one eagle. Then, as there is one eagle remaining to be divided, we reduce this to dollars, the next order on the right, and connecting the amount, 10 dollars, with the 1, we have 11 dollars. We then say, 6 in 11, *once* and 5 over; we write 1 under 1, which shows that each person will receive one dollar. Then, as there are 5 dollars remaining to be divided, we reduce these to dimes, and, connecting the amount 50 with the 4 dimes on the right, we have 54 dimes. We then say, 6 in 54 *nine* times, and write 9 under 4. This 9 shows that each person will receive 9 dimes. Then, as there is no remainder, we proceed to the next order; and, as 6 is not contained in the 3 cents, we say 6 in 3, *no* times and 3 over: we write 0 under 3, which 0 shows that there are no units of this order in the share of each person. We then reduce the 3 cents to mills; the amount 30, together with the 6 mills, makes 36 mills: we then say, 6 in 36, *six* times, and, writing 6 underneath, the operation is finished, each person's share being 11906 mills, or 1 eagle, 1 dollar, 9 dimes, and 6 mills: or (48) eleven dollars, ninety cents, and six mills. The latter is the usual method of reading.

153. As the remainder is always tens with regard to the next figure on the right, (48,) *we have only to suppose it placed on the left of that figure, and read the number as if it stood alone.*

$11160900 \div 12 = 930075$
12)11160900
930075
12
11160900 proof. (See Art. 141.)

Here, as the two first figures of the dividend are not sufficient to contain the divisor, we take three, and say, 12 in 111, *nine* times and 3 over, writing 9 under the unit figure of the part 111. We then suppose the remainder 3 to be placed on the left of the next figure 6, and say, 12 in 36, *three* times, writing 3 under 6: we then say, 12 in 0, *no* times, writing 0 under 0; also 12 in 9, *no* times, and 9 over, writing 0 under 9; then, 12 in 90, *seven* times and 6 over; and, lastly, 12 in 60, *five* times, writing the quotient figure as before.

Examples.

1. 13506 ÷ 2 =
2. 25476 ÷ 3 =
3. 21868 ÷ 4 =
4. 195425 ÷ 5 =
5. 292236 ÷ 6 =
6. 608176 ÷ 7 =
7. 608176 ÷ 8 =
8. 842256 ÷ 9 =

9. Divide 47900160 by each of the numbers 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, and find the difference between the sum of the quotients and one trillion.

Answer. Nine hundred and ninety-nine billions, nine hundred and four millions, forty-five thousand, eight hundred and eighty-eight.

154. Division is the converse or contrary of multiplication; consequently, *the one destroys the effect of the other.* Hence, a number multiplied and divided by the same number is still the same.

Thus: $8 \times 7 \div 7 = 8$ or $\frac{8 \times \cancel{7}}{\cancel{7}} = 8.$ Operation,

$$\begin{array}{r} 8 \\ \underline{7} \\ 7)\underline{56} \\ 8 \end{array}$$

For, (59 and 63,) $8 \times 7 = 7+7+7+7+7+7+7+7$: and it is evident that, from the sum of 8 sevens, we can subtract 7 eight times, which is, in substance, what we do in the division. Wherefore, the multiplication and division of a given number by one and the same number amounts to this, namely, the addition of the multiplier to itself as often as there is a unit in the multiplicand, and the subtraction of the multiplier from the sum the same number of times. Also, let us observe, that $8 \times 7 \div 7$ is the same as $8 \div 7 \times 7$. For, the first signifies the seventh part of 7 times 8, which is evidently 8; and the last 7 times the seventh part of 8, which is also evidently 8. The same reasoning will apply to all such cases: wherefore *it is of no consequence,* as to the result, which of the two operations is performed first. Lastly, as the number upon which we operate is not affected, *both operations may be spared.*

155. If we divide the product of two numbers by either of them, (60,) *the quotient will be the other.* Also, by the above, a number multiplied and divided by the same number is still the same. Therefore, having found the product of 563 × 5 = 2815, if we divide this product by 5, we shall have 563 for the quotient, thus:

$$\begin{array}{r} 5)\underline{2815} \\ 563 \end{array}$$

Here, as 2 is not sufficient, we say, 5 in 28, *five* times, (the double of the tens figure 2, plus 1, because 8 contains one more 5,) and 3 over. This 3 hundreds or 30 tens makes, with the 1 ten, 31 tens. We then say, 5 in 31, (doubling the 3,) *six* times, and 1 over. This 1 with the 5 makes 15. Then, 5 in 15, *three* times, and the work is done; the quotient being 563, as was proposed. Or thus: reading the binomial value, (49,) which is 281 tens and 5 units; we double the tens, for the number of times 5 is contained in them, which gives 562, to which adding 1, for the 5 units, we have 563, as before.

The student may observe, that the figures 1 and 3, which we carry in multiplying 563 by 5, are the same which we bring back again in dividing 2815 by 5.

156. We have seen (149) that the remainder of a division may be any number which is less than the divisor; and (146) that the division of this remaining part of the dividend is expressed by placing the divisor underneath it, with a line between: also, (145,) that this quotient, being less than a unit, is *a fraction.* The upper number, or dividend, is called *numerator,* because it shows the number of parts in the fraction; and the lower number, or divisor, is called *denominator,* because it determines the name of those parts. The quotient of any division, therefore, which has left a remainder, is rendered complete *by placing the fraction,* here described *on the right of the unit figure of that quotient.* It is plain (60) that if we multiply a fraction by its denominator, the result or product will be the numerator: this is *the same as to take away the denominator.* Wherefore, to prove by multiplication, *multiply the integral part of the quotient by the divisor, and to the product add the remainder, or numerator of the fraction,* which will give the dividend when the work is right.

The student will easily understand this by considering, that the integral part of the quotient shows the number of times the divisor has been subtracted, and that when a fraction is multiplied by its denominator the result is the numerator.

$5734999 \div 7 = 8192854\frac{4}{7}$

$$\begin{array}{r} 7\overline{)\,5734999} \\ \hline 8192854\frac{4}{7} \\ 7 \\ \hline 5734999 \end{array}$$

Note. **A number which is composed *of a whole number and a fraction* is called *a mixed number*. Thus, 5½, 6⅔, &c., are mixed numbers.**

Here, writing as usual the numbers signified by the words in italics, we say, 7 in 57, *eight* times, and 1 over: 7 in 13, *once*, and 6 over: 7 in 64, *nine* times, and 1 over: 7 in 19, *twice*, and 5 over: 7 in 59, *eight* times, and 3 over: 7 in 39, *five* times, and 4 over, and placing this 4 over 7, on the right of the unit's figure 5, the quotient is complete. To prove the work, we multiply the quotient by 7, thus: 7 times four-sevenths is 4: 7 times 5 is 35, and 4 is 39; nine and go 3, &c., as usual, and having reproduced the dividend, we consider the division correct. In the same manner the student may prove the following examples:

1. $76947385 \div 2 =$
2. $1062047953 \div 3 =$
3. $5799511287 \div 4 =$
4. $6397173134 \div 5 =$
5. $1169387459 \div 6 =$
6. $7924865731 \div 7 =$
7. $9301068145 \div 8 =$
8. $8991068357 \div 11 =$
9. $1098106577 \div 9 =$
10. $11989275575 \div 12 =$

157. When the divisor consists of several figures, we seek how often its left-hand figure is contained in the left-hand figure or two left-hand figures of the partial dividend. Also, *we increase by a unit the left-hand figure of the divisor when the next figure on the right of it is very large.* Having found the quotient figure, we multiply the divisor by it, placing the product in order as we find it under the partial dividend; and, having performed the subtraction, we bring down, to the right of the remainder, the next figure of the principal dividend; because, with regard to such figure, the remainder (46, 48, and 54) is always *a number of tens.* Having thus formed a new partial dividend, we proceed as before, and continue thus till the operation is finished. The student should attentively observe, that, if the remainder equals or exceeds the divisor, *the quotient figure is too small.* Also, if the product of the divisor and quotient figure exceeds the partial dividend, *the quotient figure is too great.*

$$159529 \div 43 = 3709\tfrac{42}{43}$$

```
                    Dividend.
Divisor 43) 159529 (3709 42/43 quotient.
            129
            ---
             305
             301
             ---
               429
               387
               ---
                42 remainder.
```

Taking 159 to contain 43, we say, 4 in 15, *three* times, and write 3 in the quotient. Because 159 is thousands, the quotient 3 is also thousands, and requires three figures on the right of it; that is, as many as remain in the dividend. We then multiply 43 by 3, saying, 3 times 3 is 9, which we write under the 9 units of 159: then 3 times 4 is 12, which we write under 15. We then subtract, and have 30 thousand for remainder. To the right of this, we bring down the figure 5 of the dividend, which is 5 hundred, and have 305 hundred for the partial dividend. We then say, 4 in 30, *seven* times, and place 7, which is 7 hundred, in the quotient. We then multiply 43 by 7, as usual, subtract, and have 4 hundred for remainder, to the right of which we bring down the 2 tens of the dividend, and have 42 tens for the partial dividend. Now, as this does not contain the divisor, we write 0 in the quotient, which shows that there are no tens, and bringing down 9, the last figure of the dividend, we say, 4 in 42, *nine* times, and write 9 in the quotient. Having multiplied the divisor 43 by this 9, and subtracted the product 387 from the last partial dividend 429, we have 42 for the final remainder of the division. We express the division of this remainder by writing it over the divisor, thus, $\frac{42}{43}$. This fraction we place in the quotient on the right of the units figure 9, and the work is complete—the quotient being $3709\frac{42}{43}$.

158. We may consider the quotient $3709\frac{42}{43}$ as the result of the equal division of 159529 dollars amongst 43 persons, in which case, each person will receive 3709 dollars and 42 of the parts of which one dollar contains 43; or (145) the forty-third part of 42 dollars, which, evidently, is nearly one dollar more. The value of this fraction, in the lower denominations of Federal Money, we can easily find (152) by continuing the division, thus:

```
43) 159529 (3709,976 +
    129
    ---
     305
     301
     ---
       429
       387
       ---
        420
        387
        ---
         330
         301
         ---
          290
          258
          ---
           32
```

Having found the quotient 3709, and the remainder 42, as before, we place a comma on the right of 3709, to show that the division, in whole numbers, is no longer possible, and to distinguish these whole units from the lower denominations which are to follow. We then place a cipher on the right of the 42 remaining dollars, which, in multiplying this number by ten, reduces it to *tenths*, or *dimes*. We then say, 4 in 42, *nine* times, and place 9 in the quotient. Now as the preceding figure is 9, it is plain that we shall have the same product as in the preceding division; we, therefore, copy the number 387, and, having placed it under 420, and subtracted, we have 33 *dimes* or *tenths* for remainder. Placing a cipher on the right of 33, which reduces these tenths to *hundredths* or *cents*, we say, 4 in 33, *seven* times; for, though 4 is contained in 33, really 8 times, and 1 over, it is easy to see that the unit figure 3 is not contained 8 times in 10, the value of the remaining 1. We, therefore, place only 7, which is 7 cents, in the quotient, and write the product 301 (which we may copy from the second division) under 330. Having subtracted as usual, and found 29 *hundredths* or *cents* for remainder, we place 0 on the right, and have 290 *thousandths* or *mills*. We then say, 4 in 29, *six* times, and place 6, which is 6 *mills*, in the quotient. Having multiplied and subtracted, we have 32 *mills* for remainder.

By placing 0, as usual, on the right of 32, which would reduce it to *ten-thousandths*, we easily see that the quotient could be extended; that the next quotient figure would be *ten-*

thousandths, the next *hundred thousandths*, &c., the generic unit of each new order being *ten times less than that of the preceding one*. Also, because this generic unit (37) is greater than any definite number of figures which can stand on the right of it, it is evident that the approximation of the quotient, actually obtained, to the true quotient, is always *within a unit of the lowest order of the quotient obtained.* The quotient 3709,976 +, which, as an abstract number, (18,) is read, three thousand, seven hundred and nine, and nine hundred and seventy-six thousandths; and, as the result of our present discussion, 3709 dollars 97 cents and 6 mills is, therefore, within *one thousandth* or *one mill* of the true quotient. We place the sign + on the right of the quotient *to signify that it is not complete.*

If we continue the operation we shall obtain the quotient 3709,976744186046511627906, with a remainder of 42. Now, as it is from the division of this remainder by 43, that we have obtained the orders on the right of the comma, it is evident that, in continuing the operation, the same figures will recur again, until we again find a remainder of 42; consequently, though we can approximate within any assignable quantity, *we never can obtain an exact quotient.*

159. The orders on the right of the comma are called *decimals;* and, as a definite number on the right of the comma (37) cannot equal a unit, such number is called *a decimal fraction.* The number ,976 +, &c. is, therefore, a decimal fraction; and, as it is the result of the actual division of 42 by 43, is the approximate value of the vulgar or common fraction $\frac{42}{43}$. The operation is called *reducing the vulgar fraction* $\frac{42}{43}$ *to a decimal.*

160. If on rejecting the right-hand figure of a partial dividend, which has one more figure than the divisor, the part on the left very nearly, though not quite, contains the divisor, as the resumption of the figure rejected implies (49) the multiplication of the part on the left by ten, as well as the addition of the figure resumed, we may confidently place 9 in the quotient. Again, if the part on the left very little exceeds one-half the divisor, we may confidently place 5 in the quotient; because the half of any number, considered as tens, is 5 times that number, considered as units.

Hence the method now very generally practised of multiplying by 5, by supposing 0 on the right and taking half the given number.

$$1593290670 \div 1594 = 999555$$

$$\begin{array}{r}
1594)\,1593290670\,(999555 \\
14346 \\
\hline
15869 \\
14346 \\
\hline
15230 \\
14346 \\
\hline
8846 \\
7970 \\
\hline
8767 \\
7970 \\
\hline
7970 \\
7970 \\
\hline
****
\end{array}$$

Proof by Multiplication.

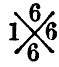
$$\begin{array}{r}
999555 \\
1594 \\
\hline
3998220 \\
8995995 \\
14993325 \\
\hline
1593290670
\end{array}$$

Proof by Division.

$$\begin{array}{r}
999555)\,1593290670\,(1594 \\
999555 \\
\hline
5937356 \\
4997775 \\
\hline
9395817 \\
8995995 \\
\hline
3998220 \\
3998220 \\
\hline
****
\end{array}$$

Here, in each partial dividend, rejecting the units figure, the part on the left, in the first three, very nearly contains the divisor; and, in two of the last three very little exceeds its half.

161. Where a division has left no remainder, as in the above example, the dividend being the product, and the divisor and quotient the two factors, the correctness of the operation may be proved by casting the nines out of these, precisely as in multiplication. Thus, casting the nines out of the divisor and quotient, we have the remainders 1 and 6 respectively. Having placed these in the side angles and their product 6 in the upper angle, we cast the nines out of the dividend, and as we find 6, which agrees with the upper figure, we conclude that the work is right.

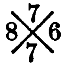


When the division has left a remainder, we cast the nines out of it, and add what remains to the figure of the upper angle, casting the nines out of the sum when it exceeds 9. The reason for proceeding thus, is, that the product of the divisor and quotient (such as we here consider the quotient) does not equal the dividend without the addition of the remainder.

$$178621 \div 197 = 906\tfrac{139}{197}$$

$$\begin{array}{r}
197)\,178621\,(906 \\
1773 \\
\hline
1321 \\
1182 \\
\hline
139
\end{array}$$

Proof.

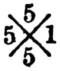


Here, after finding the numbers 8 and 6, we cast the nines out of 48, their product, and 3 remains. We also cast the nines out of the remainder 139, and 4 remains. We then add 3 and 4 together, and place the sum 7 in the upper angle, after which we cast the nines out of the dividend, and as the remainder 7 agrees with the figure in the upper angle, the work is supposed to be right. We may place the product 48 by the side of the remainder 139, and cast the nines out of the figures 48139, as though they formed but one number, which is more expeditious.

162. As there must be (151) just as many figures on the right of the first quotient figure as there are in the dividend on the right of the part first divided, we easily see that when the first partial dividend contains just as many figures as the divisor, *the divisor and quotient together will contain one figure more than the dividend;* and when the first partial dividend contains one figure more than the divisor, *the divisor and quotient together will contain just as many as the dividend.*

When the dividend is large, we put a dot under each figure as we bring it down, to avoid mistake.

1. $31406 \div 41 = 766$ and $\frac{31406}{766} = 41$



Also $41 \times 766 = 31406$

The student may prove the following examples by multiplication, by division, and by casting out the nines :

2. $239850 \div 738 =$
3. $463088 \div 412 =$
4. $7483566 \div 82 =$
5. $27413568 \div 523 =$
6. $\frac{47171718}{627} =$
7. $71710048 \div 796 =$
8. $\frac{70469217}{819} =$
9. $23159220 \div 2345 =$

163. When the second figure of the divisor, counting from the left, is very large, in seeking how often such divisor is contained, as directed, (157,) *the quotient cannot be too great.*

Numerator / Denominator $\frac{2701}{397} = 6\frac{319}{397}$

$$\begin{array}{rl} & \text{Dividend} \\ \text{Divisor } 397) & 2701\ (6\tfrac{319}{397} \text{ quotient.} \\ & \underline{2382} \\ & 319 \text{ remainder.} \end{array}$$

Here, instead of saying 3 in 27, we say 4 in 27, (because the divisor is much nearer to 400 than to 300,) and find 6, which is the true quotient figure, instead of which, we should have found 9, and should consequently, have been obliged to make three useless trials. We easily perceive that this can never give too great a quotient, for if it could, then might 400 be contained in the dividend a greater number of times than 397, which is impossible,

Because, $\frac{319}{397} \times 397 = 319$, (60 and 156,) the above example is proved thus: $6 \times 397 + 319 = 2701$.

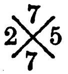
This expression is the same as to say, *multiply the divisor and the integral part of the quotient together, and to the product add the remainder or numerator of the fractional part,* which will give the dividend. The student may, in like manner, prove all similar divisions: also (161) by casting out the nines.

164. It is sometimes so nice a point to determine the true quotient figure, that most persons, however skilful, are liable to err; but the multiplication of the divisor by the quotient figure immediately shows when this figure is too great, because, in this case, *the product will be greater than the partial dividend from which it is to be subtracted.* Also, when the subtraction of the product from the partial dividend leaves a remainder equal to or greater than the divisor, as the divisor still contained in this remainder, *the quotient figure is too small.*

$6814834 \div 113582 = 59{,}999 +$

113582) 6814834 (59,999 +
tens 567910
units { 1135734
1022238
tenths { 1134960
1022238
hundredths { 1127220
1022238
thousandths { 1049820
1022238
27582 thousandths.

Proof by 9.



See article 161.

We here first say 1 in 6, or rather 11 in 68, *six* times, and place 6 in the quotient; but, having multiplied the divisor by this figure and found that the product 681492 is greater than the partial dividend 681483, and consequently the figure 6 too great, we efface it, and, having substituted the figure 5, proceed as usual.

Or thus, taking 6 for the first quotient figure, say 6 times 11 is 66, which taken from 68, leaves 2. This 2, with the next figure, makes 21. Then 6 times 3 is 18, which taken from 21, leaves 3. This, with 4, makes 34. Then 6 times 5 is 30, and 30 from 34 leaves 4. This, with 8, makes 48. Then 6 times 8 is 48, and 48 from 48, leaves 0. Lastly, as 6 times 2 is not contained in 3, we see that 6 is too great, and put 5 in its stead as before.

Proof by multiplication.

$$\begin{array}{rl} 113582 & \\ 59999 & \text{thousandths} \\ \hline 1135706418 & \\ 567910 & \\ \hline 6814806418 & \text{thousandths} \\ 27582 & \text{thousandths} \\ \hline 6814834{,}000 & \end{array}$$

Here it is plain that the product, formed by repeating 59999 thousandths 113582 times, is a number of thousandths, as well as the remainder 27582. Hence, the sum of these, which is 6814834000, is also a number of thousandths; wherefore, to obtain the units in this number, (144,) we divide by 1000, which is done (56) by cutting off the three ciphers on the right, the number on the left being the dividend, as required.

1. $\frac{728976}{803} = 907\frac{655}{803}$, and $\frac{728976}{907} = 803\frac{655}{907}$
2. $27065849 \div 356 =$
3. $\frac{26665942}{886} =$
4. $3141593 \div 7854 = 399{,}999 +$
5. $78539992146 \div 7854 =$
6. $18437688851569 \div 4293913 =$
7. $111030011 \div 199 = 557939{,}75 +$
8. $\frac{3938725}{69} = 57082{,}971 +$
9. $12167395 \div 1219 = 9981{,}456 +$
10. $78987116249 \div 19818 = 3985624{,}9999 +$
11. $\frac{612339}{99999} = 6{,}1234512345$, &c.
12. $\frac{5684973 \times 285311670611}{25937424601} = 62534703$

165. The greater the dividend, or number to be divided into

equal parts, the greater the quotient, or value of each part. Therefore, *to multiply the dividend* (150) *is to multiply the the quotient;* and, inversely, *to divide the dividend is to divide the quotient.* Again, the greater the divisor, or number of equal parts, the less the quotient, or value of each. Therefore, *to multiply the divisor is to divide the quotient;* and inversely, *to divide the divisor is to multiply the quotient;* that is to say, the operation performed upon the dividend produces upon the quotient *the same effect;* but that performed upon the divisor, *the contrary.* But (154) multiplication and division destroy the effect of each other; wherefore, to multiply or divide both dividend and divisor by the same number has *no effect* upon the quotient.

$$\begin{array}{l} \text{dividend} \\ \text{divisor} \end{array} \frac{6}{16} = 0{,}375 \text{ quotient.}$$

$$\begin{array}{l} \text{dividend mult. by 2} \\ \text{same divisor -} \end{array} \frac{6\times2}{16} = \frac{12}{16} = 0{,}75 = \overset{\text{quotient mult. by 2.}}{0{,}375\times2}$$

$$\begin{array}{l} \text{dividend div. by 2} \\ \text{same divisor -} \end{array} \frac{6\div2}{16} = \frac{3}{16} = 0{,}1875 = \overset{\text{quotient div. by 2.}}{0{,}375\div2}$$

$$\begin{array}{l} \text{dividend unaltered} \\ \text{divisor mult. by 2} \end{array} \frac{6}{16\times2} = 0{,}1875 = \overset{\text{quotient div. by 2.}}{0{,}375\div2}$$

$$\begin{array}{l} \text{same dividend -} \\ \text{divisor div. by 2} \end{array} \frac{6}{16\times2} = \frac{6}{8} = 0{,}75 = \overset{\text{quotient mult. by 2.}}{0{,}375\times2}$$

$$\begin{array}{l} \text{dividend and divisor} \\ \text{both mult. by 2 -} \end{array} \frac{6\times2}{16\times2} = \frac{12}{32} = \overset{\text{quotient unaltered.}}{0{,}375}$$

$$\begin{array}{l} \text{dividend and divisor} \\ \text{both div. by 2 -} \end{array} \frac{6\div2}{16\div2} = \frac{3}{8} = 0{,}375 \text{ quotient the same.}$$

166. The decimal value of the generic unit in each of the fractions $\frac{3}{8}$, $\frac{6}{16}$, and $\frac{12}{32}$, is found by dividing a unit by the denominator, and that of the fraction by multiplying the value of the generic unit by the numerator, thus:

$$\frac{1}{8} = 0{,}125 \text{ and } \frac{3}{8} = 0{,}125\times3 = 0{,}375$$
$$\frac{1}{16} = 0{,}0625 \text{ and } \frac{6}{16} = 0{,}0625\times6 = 0{,}3750$$
$$\frac{1}{32} = 0{,}03125 \text{ and } \frac{12}{32} = 0{,}03125\times12 = 0{,}37500$$

The ciphers on the right of the numbers 0,3750 and 0,37500 may be omitted, as having no effect upon the value of the number 0,375, because (165) $\frac{375}{1000} = \frac{3750}{10000} = \frac{37500}{100000}$, &c.

167. When any two numbers are each multiplied by the same number, the two products are called *equimultiples* of the two numbers. Thus, if we multiply the two terms of the fraction $\frac{3}{8}$ each by 2, the resulting terms 6 and 16 are equimultiples of 3 and 8. But we have seen (165) that the quotient $\frac{3}{8}$ is the same as the quotient $\frac{6}{16}$ or $\frac{12}{32}$, &c. Wherefore, if any equimultiples whatever be taken of the terms of a fraction, the resulting fraction is still of the same value. Hence, it is evident that the same fraction may be represented by an infinity of different numbers, seeing that the two terms may be multiplied by each of the numbers in the natural scale 2, 3, 4, 5, 6, 7, &c., ad. inf. Thus, $\frac{3}{8} = \frac{6}{16} = \frac{9}{24} = \frac{12}{32} = \frac{15}{40}$ $\frac{18}{48} = \frac{21}{56}$, &c., ad. inf.

The least numbers by which a fraction can be expressed are called *its lowest terms.* Thus, the terms 3 and 8 are the lowest terms of the above fraction. Each of the other expressions, $\frac{6}{16}$, $\frac{9}{24}$, $\frac{12}{32}$, &c., is called *a value of* $\frac{3}{8}$.

We may further elucidate this by the smaller Spanish coins or parts of a dollar, namely, halves, quarters, eighths, and sixteenths, which, though they do not accord with the decimal Federal Money of the United States, are still in general use.

Thus, $\frac{1}{8} = \frac{2}{16}$
$\frac{1}{4} = \frac{2}{8} = \frac{4}{16}$
$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$

It will also be useful to the student to render himself betimes familiar with the following decimal values of these coins, thus:

1. $\frac{1}{2}$ = ,50 or 50 cents
2. $\frac{1}{4}$ = ,25 or 25 cts.
3. $\frac{3}{4}$ = ,75 or 75 cts.
4. $\frac{1}{8}$ = ,125 or 12½ cts.
5. $\frac{3}{8}$ = ,375 or 37½ cts.
6. $\frac{5}{8}$ = ,625 or 62½ cts.
7. $\frac{7}{8}$ = ,875 or 87½ cts.
8. $\frac{1}{16}$ = ,0625 or 6¼ cts.
9. $\frac{3}{16}$ = ,1875 or 18¾ cts.
10. $\frac{5}{16}$ = ,3125 or 31¼ cts.
11. $\frac{7}{16}$ = ,4375 or 43¾ cts.
12. $\frac{9}{16}$ = ,5625 or 56¼ cts.
13. $\frac{11}{16}$ = ,6875 or 68¾ cts.
14. $\frac{13}{16}$ = ,8125 or 81¼ cts.
15. $\frac{15}{16}$ = ,9375 or 93¾ cts.

168. When a fraction is expressed by equimultiples of its lowest terms, it is often necessary, in order to expedite calculation, to find those terms; for, if we would multiply a number by 122936 and divide the product by 138303, which is the same as to multiply the number by the fraction $\frac{122936}{138303}$, we should obtain the result much more readily if we could conveniently discover that $\frac{122936}{138303} = \frac{8}{9}$, in which case we should simply multiply by 8 and divide by 9 or, subtract from the number one-ninth of itself. This we shall illustrate thus:

$$\frac{428695731 \times 122936}{138303} = 381062872$$

First method 428695731
122936

3858261579	nine hundred times
15433046316	thirty-six times
857391462	two thousand times
5144348772	120 thousand times

138303) 52702138386216 (381062872
414909
1121123
1106424
146998
138303
869538
829818
397206
276606
1206002
1106424
995781
968121
276606
276606
******

Second method.

428695731
8
9) 3429565848
381062872

Third method.

428695731
one-ninth - 47632859
eight-ninths 381062872

To facilitate the research above alluded to, as well as for other important purposes, the properties of numbers, with which we commence the succeeding book, will be found of great utility.

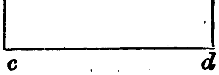
BOOK II.

OF THE SQUARE, CUBE, AND PARALLELEPIPED—POWERS OF NUMBERS—PRIME NUMBERS—PROPERTIES OF NUMBERS—PRIME FACTORS—LEAST COMMON MULTIPLE—GREATEST COMMON MEASURE, RATIO, &c.—ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF VULGAR FRACTIONS AND MIXED NUMBERS; ALSO DIVISION BY FACTORS.

SECTION VII.

OF THE SQUARE, CUBE, AND PARALLELEPIPED—POWERS OF NUMBERS—PRIME NUMBERS—PROPERTIES OF NUMBERS.

169. **Because** a number multiplied by a unit (57) is still the same, a unit multiplied by a unit is still a unit. But (108) the product of two equal factors represents a square: wherefore, the unit which is the product of 1 × 1 represents a square. For example, let 1 represent the line *a*————*b*; then *ab* × *ab* or 1 × 1 will give the content of the figure *a b c d*, each of the factors being the length of one side.



170. We have seen (110) that a superfices, or surface, is measured by multiplying together its length and breadth. A superfices is, therefore, called *a quantity of two dimensions.*

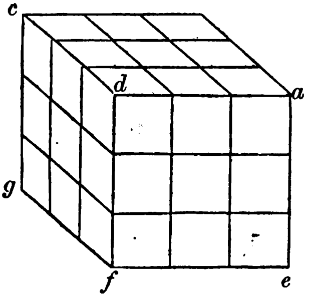
A solid is that which has length, breadth, and thickness, and is, therefore, called *a quantity of three dimensions.*

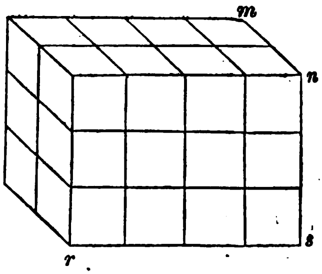
A cube is a solid body having its length, breadth, and thickness all equal, and is enclosed by six equal squares.

Other figures of six sides, the opposite sides of which are equal parallelograms, are called *parallelepipeds.*

The content of a solid is estimated *in cubes,* and is thus found: Multiply the length, taken in feet, inches, or any other measure, by the breadth, taken in the same measure, and this product by the depth* or thickness, taken also in the same measure. The last product will be the number of cubic feet or cubic inches (or whatever measure you employed) contained in the figure.

When the solid is a cube, as all its sides are equal and all its angles right angles, *the length of one side only is sufficient* to find its content.





Let the figure *a b c d e f g* represent a cubic yard. Then, as $ab = bc = ae$, each being one yard long, $ab \times bc \times ae$ is the same as $bc \times bc \times bc$ or $1 \times 1 \times 1 = 1$, the figure 1 on the right being *the cube itself* or solid body *a b c d e f g*, and each unit in the expression $1 \times 1 \times 1$ being *the length of one of the equal sides*, or *bc*.

The reason of the rule for finding the content will appear thus: in multiplying the length by the breadth, we obtain (110) *the number of squares in the upper surface;* and, it is evident, that for every unit in depth, we shall have *as many cubes as there are squares in this surface,* this number of cubes will, therefore, be repeated *as often as there is a unit in the depth,* which (59) is the same as *to multiply by the depth.* Thus, in the above figures, referring each of the smaller divisions to 1 foot, we see that the cubic yard contains 27 cubic feet and the parallelepiped 24.

Hence every number which is the product of three factors may be considered as representing *a cube* or *parallelepiped:* namely, if the factors be all equal, *a cube;* if unequal, *a parallelepiped.*

When we say that a number represents a geometrical figure, it must be understood that the same number may represent

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*** The student must understand, that we here speak of a regular solid of six sides, and that when we speak of breadth and depth, we mean perpendicular breadth and depth.**

the content of a variety of geometrical figures, not only of such as are here mentioned, but of others, of which the object and limits of this work do not permit us to treat. For example, the number 64 may express the content of a square, of a cube, and of differently-shaped parallelograms and parallelepipeds.

171. The *square* and *cube* of a number—evidently so called from the nature of the geometrical figures which they represent—generally retain those names, though they are also sometimes called *the second* and *third powers* of that number, the number itself being *the first power.*

Because a solid has only 3 dimensions, powers higher than the cube are *not considered as representing geometrical figures.*

From the number of times that a number is factor in its continual multiplication, the several products, after the cube, are respectively called *the fourth, fifth, sixth, seventh,* &c. *powers* of that number.

A small figure, called *the index*, showing the number of times that a number is factor in any of its powers, is placed on the right, thus: 2^2, 2^3, 2^4, 2^5, signify *the square, cube, fourth, and fifth powers of* 2, instead of 2×2; $2\times2\times2$; $2\times2\times2\times2$; $2\times2\times2\times2\times2$, and are read, 2 squared, 2 cubed, 2 fourth, 2 fifth, and similar expressions in like manner.

As the index shows how often the number is factor, if the indices of two or more powers of the same number be added, the sum will be the index of the power to which the number is raised by multiplying those powers together. Thus $3^2\times3^3=3^5$; $2^3\times2^4=2^7$.

172. If a number will divide the whole of another number and one of its parts, *it will also divide the other part.*

Let B be any number, and C, one of its parts; also, let $B-C=D$: then, if A be any number that will divide B and C, it will also divide D. For, as B is a multiple of A, $B=A+A+A$, &c., A being taken a certain number of times. Again, because C is a multiple of A, $C=A+A$, &c., A being taken a certain number of times: wherefore D, which is the difference between B and C, must be a certain number of times A,—that is, A or its multiple,—and, therefore, A will divide or measure D. Hence, if a number divides the greater of two numbers and the less, *it will divide their difference.*

173. It has been shown in the preceding article, that any number that will divide B and C will divide their difference D; but no number greater than D can be contained in D; therefore, if D, the difference, between B and C, will divide both,

it is the greatest number that will divide both. It is also plain that, if any number will divide both, D is that number, or a multiple of that number. Wherefore, the difference between two numbers is the greatest number that can divide both without a remainder: and, if any number will divide each of two numbers without remainder, *their difference is that number or its multiple.*

174. A number which cannot, without remainder, be divided by any other number, is called *a prime number*, or simply *a prime.*

In the regular formation of numbers, such numbers must arise; for, as one number cannot contain another greater than itself, it is plain that 2 is the first prime number, and that the number 3, which is odd, and, consequently, not divisible by 2, is also a prime number.

PRIMES.	SQUARES.	CUBES.
2	4	8
3	9	27
5	25	125
7	49	343
11	121	1331
13	169	2197
17	289	4913
19	361	6859
23	529	12167
29	841	24389
31	961	29791
37	1369	50653
41	1681	68921
43	1849	79507
47	2209	103823
53	2809	148877
59	3481	205379
61	3721	226981
67	4489	300763
71	5041	357911
73	5329	389017
79	6241	493039
83	6889	571787
89	7921	704969
97	9409	912673

Numbers which are not prime are called *composite numbers; because each is composed by the multiplication or involution of two or more primes.* Wherefore, because they can be divided by 2, *all even numbers except* 2 *are composite* and terminate *with* 0, 2, 4, 6, or 8. Hence, *every prime number, except* 2, *is odd;* and, when it consists of more than one figure, terminates *with* 1, 3, 7 or 9.

The square of any prime number, except 2 and 5, terminates *with* 1 or 9.

When a prime ends with 1, *so do all its powers.*

When it ends with 9, *the square ends with* 1. When with 3 or 7, *the square ends with* 9.

When either a prime or its cube ends with 3, *the other ends with* 7.

When either ends with 1 or 9, *the other ends with the same figure.*

The primes which succeed 97, are 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 287, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 779, 787, 791, 793, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997, 1009, 1013, 1019, 1021, 1031, 1033, 1039, 1049, 1051, 1061, 1063, &c.

To determine whether a number ending with 1, 3, 7, or 9 is or is not prime, *divide the number by each of the prime numbers* 3, 7, 11, 13, 17, &c., *till the quotient is not greater than the divisor.* When every division within this limit gives a remainder, *the given number is prime:* because any greater divisor would give a less quotient, and this would fall within the range of numbers already tried. Thus the scholar will find that 1067 is a multiple of 11 and 97: also that 1069 is a prime number.

Suppose we would examine 14363. Having tried the numbers 3, 7, 11, 13, 17, 19, without success, we say that, as the number ends with 3, it is probably a multiple of some one of the higher primes ending with 3. We, therefore, try 23, 43, &c., thus:

14363	14363	14363
23	43	53
1434	1432	1431
184 (8·23)	172 (4·43)	371 (7·53)
125	126	106
115 (5·23)	86 (2·43)	106 (2·53)
10	4	***

By the remainder at the end of each of the two first of these operations, we see that neither 23 nor 43 is a measure of 14363. But, as the third operation gives no remainder, we find that 53 is a measure.

Numbers which have no common factor or measure are said to be *prime to each other.* Thus, 10, 21, and 121 are prime to each other.

If 53 measures 14363, the remainder, 1431 tens, is (172) a multiple of 53. But the product, 371 tens, is a multiple: therefore, the remainder, 106 hundreds, is also a multiple. Now we have subtracted 53 times 1, 53 times 7 tens and 53 times 2 hundreds, and have no remainder. Hence, the third operation shows that 14363 is a multiple of the primes 53 and 271.

In thus operating, we always multiply the prime, if required, *by such a number as will render its unit figure the same as that of the number from which we subtract.*

472903
23 once
47288
138 (6 tens)
4715
115 (5 hundreds)
46
46 (20 thousands)
**

20561
261 (29 × 9)
203
203 (29 × 7 hundred)
***
709 a prime number

This operation shows that 472903 is the product of the 3 primes 23, 29, and 709.

We must here observe, that we cannot always thus easily obtain satisfactory results, but sometimes find this research exceedingly laborious.

175. The following properties of numbers will be found exceedingly useful *in reducing fractions to their lowest terms, in resolving numbers into their prime factors,* &c., by which, as well as by other means, they serve greatly to abridge the labour of calculation.

1. Even numbers are *divisible by* 2.

2. When the sum of the figures of a number is divisible by 3, *the number is divisible by* 3.

3. Because 4 measures 100, when 4 divides the number expressed by the two right-hand figures of a given number, *the given number is divisible by* 4.

4. Because 10 is a multiple of 5, and 5 contains itself, when the right-hand figure of a number is 5 or 0, *the number is divisible by* 5.

5. When an even number is divisible by 3, *it is divisible by* 6.

6. Because $3.7 = 21$, and $7.13 = 91$, when the unit figure is $\frac{1}{2}$, or $\frac{1}{9}$ of the part on the left, *the number is divisible by* 7. Again, because $7.15 = 105$, and $7.43 = 301$, when the number expressed by the two right-hand figures is 5 times the part on the left, or $\frac{1}{3}$ of it, 7 *will measure.* Also, in the latter case, 43 *will measure.*

7. Because 8 measures 1000, when 8 measures the number expressed by the three right-hand figures of a number, 8 *measures the number.*

8. When the sum of the figures of a number is divisible by 9, *the number is divisible by* 9, (113.) Also, the difference between a number and the same number reversed, *is, when not a cypher, divisible by* 9. Thus, $8176 - 6718 = 1458$, which is a multiple of 9. For, in casting the nines out of each number, we have the same remainder; and, by the subtraction, *the remainder is cancelled or destroyed.* Hence, in accounts, an error, divisible by 9, may suggest *that a number has, by mistake, been reversed.*

9. Because $7.11.13 = 1001$, the numbers 2002, 3003, 4004, &c. to 9009, included: also, 51051, 63063, 324324, 561561, &c., which are easily recognized *as multiples of* 1001, are divisible *by* 7, 11, *and* 13. Also, when the difference between the numbers in the two periods is 7, 11, or 13, or a multiple of any one of them, *the number is a multiple of that factor accordingly.* Thus 561554 is a multiple of 7: 561539 is a multiple of 11; and 561522 is a multiple of 13.

10. Because $3.17 = 51$, $6.17 = 102$, and $17.59 = 1003$, *the prime* 17 *will measure* a number; when the unit figure is $\frac{1}{5}$ of the part on the left, or when the number expressed by the two right-hand figures is double the part on the left; also when the number in the first period is 3 times the part on the left, 17 *and* 59 *will measure.* Thus, 459 and 2448 are multiples of 17, and 157471 is a multiple of 17 and 59.

11. Because $19.53 = 1007$, *either* 19 *or* 53 *will measure* a number, when the number in the first period is 7 times the part on the left of that period. Thus, 12084 and 113791 are multiples of 19 and 53.

12. Because 1009 is prime, when the number in the first period is 9 times the part on the left, *the factors of this part and* 1009 *are the factors of the number.* Thus, in the number 91819, the factors of 91, which are 7 and 13 and 1009 are the only factors. In like manner, when the number in the first period is 13, 19, 21, 31, 33, 39, 49, 51, 61, or 63 times

that in the second, *we should recognize the number as a multiple of* 1013, 1019, 1021, 1031, 1033, &c., *respectively, each of which is a prime number.* Thus, in the number 22682, we recognize the factors 2, 11, and 1031.

176. The binomial values of the alts 100, 10000, 1000000, &c.; that is, of those having an even number of ciphers on the right, are 99+1; 9999+1; 999999+1, &c., the nines of each being divisible by 11. Therefore, if we divide each of these alts by 11, the remainder will be 1: consequently, if we divide an alt figure in any of the odd places by 11, *the remainder will be that figure.* Hence the figures of a number, in the odd places, may be considered *as remainders of so many single units,* the value of each having been divided by 11.

The alt 10 is (11 — 1): hence 40 is (11 — 1)4: or 11 + 11 + 11 + 11 — 4; or 11 + 11 + 11 + 7: that is, in dividing the alt 4 by 11, the remainder is *its undecimal complement* 7, or *what it lacks of* 11.

If we divide 4000 by 11, the first division, that of 40 hundreds, gives 7 for remainder; the next, of 70 tens, gives 4; and the last, of 40, gives 7, as at first. Hence, the division of any figure in an even place by 11, gives for remainder *the undec. comp. of that figure.* The figures of a number in the even places are, therefore, *the undec. comps. of the remainders that would arise in dividing the value of each by* 11.

177. If, from a number, we subtract the sum of the figures in the odd places, and, to the remainder, add the sum of the figures in the even places, the result will be *a multiple of* 11. For, we thus take away *the remainders that would arise in dividing, by* 11, *the value of the figures in the odd places:* and add *the complements of the remainders that we should find in dividing the value of the figures in the even places;* and it is plain that the sum of these remainders and their comps. is a multiple of 11.

Wherefore, because addition and subtraction destroy each other, when the sum of the figures in the even places is equal to that of those in the odd places: or, when the difference between these sums is 11, or a multiple of 11, *the number is a multiple of* 11. The numbers 121, 341, 297, 792, 3454, 18579, 73986, 87969871 are therefore divisible by 11.

As the figures in the odd places may be considered remainders, it is plain that, when their sum is 11, or a multiple of 11, and there are ciphers in the even places, the number is divisible by 11. Thus, the numbers 209, 902, 308, 803, 605, 506, 30602, 20306, 20603, 60203, 60302, 90805, 9090807, 8070909, &c., are multiples of 11.

Also, if we place a nought on the right of each of these numbers, as this will not affect the exactness of the division, we say that, when the sum of the figures in the even places is 11, or a multiple of 11, and there are ciphers in the odd places, *the number is divisible by* 11. The numbers 2090, 9020, 3080, &c., are therefore multiples of 11.

From the above it is obvious that, when a number is divisble by 11, *the sum of the figures in the odd places must be equal to that of those in the even places; or the difference between these sums must be* 11 *or its multiple.*

Also, when the number is not divisible by 11, the remainder will be, when the sum of the figures in odd places is the greater, *the difference between the two sums;* and, when the sum of the figures in even places is the greater, *the undec. comp. of that difference.*

When the first of these two sums is greater by more than 11, *subtract* 11 *from the difference, till the remainder is less than* 11. When the last is greater by more than 11, *subtract the remainder from the nearest multiple of* 11 *that exceeds the difference.*

Thus, in the number 92918, the first sum is $8 + 9 + 9 = 26$; the last is $1 + 2 = 3$: then $26 - 3 = 23$, the difference, and $23 - 22 = 1$, the remainder found in dividing 92918 by 11.

$$\begin{array}{r} 11) \underline{92918} \\ 8447 - 1 \end{array}$$

Having first found the remainder, we may by it find the quotient: thus we say, 1 from 8, the right-hand figure of the dividend, 7; which we write under 8: then proceeding towards the left: 7 from 11, *four,* which we write under 1. Then, having added 10, we add 1 to the 4 just written, and say, 5 from 9, *four,* which we write under 9. Then, 4 from 12, *eight;* which we write under 2. Lastly, adding 1 to 8, we say 9 from 9, *nought,* which we do not write. This is the countermarch of the method of multiplying by 11, (126).

In the number 909281, the sums change places, the last being 26, and the first 3. Having found the difference 23, we subtract it from 33, the nearest multiple of 11 that exceeds it: thus, $33 - 23 = 10$, the true remainder.

$$\begin{array}{r} 11\overline{)909281} \\ 82661 - 10 \end{array}$$

Here we say, 10 from 11, *one;* 2 from 8, *six;* 6 from 12, *six;* 7 from 9, *two;* 2 from 10, *eight.*

178. The sum found by adding a number, consisting of an even number of figures, to the same number reversed, is *a multiple of* 11. Thus 6718 + 8176 = 14894, which is a multiple of 11.

By reversing the number, *the figures in odd and even places change with each other,* so that though 6718 ÷ 11, gives a remainder of 8, 8176 ÷ 11, gives a remainder of 3, the undec. comp. of 8. Hence, the sum of the numbers, which includes the sum of these remainders, is a multiple of 11.

When the number of figures is odd, the difference between a number and the same number reversed is, *when not a cipher, divisible by* 11.

Thus, 975 — 579 = 396, a multiple of 11.

Here, in reversing the number, the figures in odd and even places do not change; consequently, in dividing each number by 11, we have the same remainder, and by subtraction, *the remainder is cancelled;* hence, the result is divisible by 11.

But the result (175, par. 8) is divisible by 9: therefore, when the number of figures is odd, *the result is divisible by* 9 *and* 11, that is, by 99.

Also, when the number of figures is odd, and the number odd, and odd also when reversed, the difference is even, and hence divisible by 2: that is, it is divisible *by* 2, 9, *and* 11, *and consequently by* 198.

When an accountant, therefore, finds in his calculation an error of 198 or its multiple, this should induce him to suspect that an odd number, consisting of an odd number of figures, and having its left-hand figure odd, has been reversed.

179. The square, cube, &c. of one prime number *cannot be a multiple of another prime number,* much less of the square, the cube, &c. of that other number. For, if possible, let 25, which is the square of five, be a multiple of 3. Multiply 3 and 5 together, then because 3 measures 25 and 15, it will also (172) measure 10, which is their difference. Also, because 3 divides 15 and 10, it will divide their difference 5, which is absurd.

Again, let 25 be a multiple of 7, which is greater than 5.

Find the product of 7 and 5, and because 7 divides 35 and 25, it will divide 10, their difference, or its multiple 20. Also because 7 divides 25 and 20, it will divide their difference 5, which is absurd. Now if the square, cube, &c. of a prime number cannot be a multiple of another prime number, it is evident that it cannot be a multiple of any of the powers of that other number. Wherefore, because the powers of different primes have no common factor, (that is to say, because no number can be found that will divide any two of them,) they are said to be *prime to each other.* Hence, the numbers 8, 9, 25, 49, &c. are prime to each other. Hence also, the squares, cubes, &c. of numbers prime to each other, are *still prime to each other.*

180. The product of two or more primes, *can never be a multiple of any other prime number.* Also, if several numbers be primes or primes to each other, *no number less than their product can be their common multiple.* First, if possible, let the product of the two primes 11 and 5 be a multiple of 7. Multiply 7 and 5 together. Then, because 7 divides 55 and 35, it will divide their difference 20, or its multiple 60. Also, because 7 divides 60 and 55, it will divide their difference 5, which is absurd.

Wherefore, if several numbers be prime to each other, the product of any two of them is prime to a third, and consequently, that of any three of them is prime to a fourth and so on: therefore, when several numbers are primes or primes to each other, *their product is their least common multiple.* Hence, it is plain that *one number cannot measure another, unless all its factors are factors of that other.*

SECTION VIII.

PRIME FACTORS—LEAST COMMON MULTIPLE—GREATEST COMMON MEASURE, RATIO, &C.

181. To find the prime factors of any composite number, divide the given number by the least prime number by which this can be done. Divide the quotient by the same prime, if possible. Continue to divide each successive quotient by the same prime till this is no longer possible. Then divide in the same manner, by the next least prime till it is exhausted, and then by the next, and so on, till the quotient is a prime num-

ber. The several divisors and last quotient will show the prime factors of the given number and the highest power of each. For example, we find the prime factors of 2520 as follows:

$$\begin{array}{r} 2\,\underline{)\,2520} \\ 2\,\underline{)\,1260} \\ 2\,\underline{)\,630} \\ 3\,\underline{)\,315} \\ 3\,\underline{)\,105} \\ 5\,\underline{)\,35} \\ 7 \end{array}$$

By which we see that $2\times2\times2\times3\times3\times5\times7$, or, which is the same, $2^3.3^2.5.7=2520$. Also, by variously multiplying the factors, we find that 2520 is a common multiple of 2, 4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 28, 30, 35, 36, 40, 42, 45, &c.

For practice, the prime factors of the following numbers may be found in like manner: 24, 64, 81, 512, 6561, and 8712.

But the student need not confine himself entirely to this routine; for, availing himself of the properties of numbers already known, he may, in many cases, abridge the operation. Thus, by adding the digits of 24, he will easily discover, that this number is composed of the factors 2^3 and 3: that 64, being $2^3\times2^3$, is 2^6; that 81, being 9^2, is necessarily 3^4, and so on.

182. It is evident, that any power of a number *is a multiple of all the inferior powers* of that number. Again, from what has been said (179 and 180) it is evident that a number which is a multiple of another number, must be a multiple of *all the prime factors of that other number*, each being taken in *the highest power* in which it is found factor in that number. Also, that when a number is a multiple of all the factors of another number, each taken in the highest power in which it is found factor, it is also a multiple of that other number.

From which we easily infer, that the least common multiple of several composite numbers is *the product of the several prime numbers which enter into these composite numbers as factors*, each being taken *only once, and in the highest power* in which it is found factor in any of the numbers. For example, to find the least common multiple of 16, 24, 48, 60, 81 and 128, we find by resolving these numbers into their prime factors, as in the preceeding article, that they are respectively 2^4; $2^3.3$; $2^4.3$; $2^2.3.5$; 3^4 and 2^7; then taking each prime in its highest power, and rejecting the others, we have $2^7\times3^4\times5$, or $128\times81\times5=51840$, the required multiple.

What is the least common multiple of the numbers 12, 18, 21, 63, 72, 114, 216, and 231?

183. When the same power of the same prime number is factor in each of two composite numbers, if we divide both by that prime number, and continue the division as often as possible, it is evident that *the number of divisions will be the index of the power.* Again, if any number of multiples, containing different powers of the same prime, be divided by that prime, and the division continued as long as the prime will divide any two of them, this division will exhaust *all the inferior powers of the prime,* and the quotient arising from the division of the multiple containing the highest power will still retain the prime, as factor, *the number of times expressed by the difference between its index in this multiple and its index in the multiple containing its next lower power.* If, therefore, this last quotient and the several divisors be multiplied together, the prime will be found factor in the product *the same number of times as in the multiple containing its highest power.* Therefore, to expunge, that is, to drive out or efface, all the inferior powers of the prime factors of several composite numbers, so as to find, by the multiplication of those factors, each in its highest power, the least common multiple of all the composite numbers, we have the following

Rule.

Write the numbers in a horizontal line; draw a line underneath, and divide as many as you can, which must be at least two of them, by the least prime number by which this can be done, placing those numbers which cannot be divided below the line, together with the quotients. Divide the numbers below the line in the same manner, by the same divisor, and continue so to divide as often as possible; that is, till the same divisor will no longer divide two of them. Having exhausted the first divisor, take for a divisor the next least prime that will divide two or more, with which proceed as with the first. Continue thus till the numbers below the line are prime to each other. Lastly, multiply the divisors and numbers below the line together, and the product is the least common multiple. When the quotient is a unit, *it need not be written.*

Example.

To find the least common multiple of 8, 9, 12, 15, 24 and 27, we place the numbers thus:

$$\begin{array}{rl} 2) & \underline{8,\ 9,\ 12,\ 15,\ 24,\ 27} \\ 2) & \underline{4,\ 9,\ \ 6,\ 15,\ 12,\ 27} \\ 2) & \underline{2,\ 9,\ \ 3,\ 15,\ \ 6,\ 27} \\ 3) & \underline{9,\ \ 3,\ 15,\ \ 3,\ 27} \\ 3) & \underline{3,\ \qquad 5,\ \qquad 9} \\ & \qquad\quad 5,\ \quad 3 \end{array}$$

$2\times2\times2\times3\times3\times3\times5=1080$, the least common multiple.

We then first divide 8, 12 and 24 by 2, and place the quotients 4, 6, and 12 respectively under the numbers divided, at the same time placing below the line, together with the quotients, the numbers 9, 15, and 27, in which 2 is not factor. We then divide 4, 6, and 12 by 2; placing the quotients 2, 3, and 6, and the numbers 9, 15, and 27 below, as before. Finding that 2 will still divide two of the numbers, we divide 2 and 6 by 2, and omitting the quotient 1, we place the quotient 3 and the numbers 9, 3, 15, and 27 below, as before. Then, as 2 will no longer divide two of the numbers, and as they are all divisible by 3, we divide all by 3, and place the quotients 3, 5, and 9 below the line. Again, as 3 and 9 are divisible by 3, we divide and place the quotient 3 and the prime number 5 below the line. Lastly, as 3 and 5 are prime, we multiply these and all the divisors together, and have 1080 for the least common multiple.

By this method, we have amongst the divisors all the prime factors which are common to any two of the given numbers, and each is repeated as often as it is common factor in any two. Also, if any of these is still factor in one of the numbers below the line, that number is still retained, and is never again divided by the same factor. Therefore, each prime which is several times factor in any of the given numbers, will be just as many times factor in the multiple as in that number in which it is oftenest factor.

Again, as we have below the line all the numbers, whether prime or not, which have no common factor, it is evident that when we multiply these and the divisors together, the product is the product of all the prime factors of the given numbers, each taken in the highest power in which it is found in any one of them. Wherefore this product is a common multiple of all the given numbers.

Now as no prime number is found factor amongst the numbers multiplied oftener than in some one of the given numbers: also, (180,) as no number can be a multiple of another, unless all the prime factors of that other enter into it as factors, it is evident that the product is the least common multiple. For, if we omit any one of the prime factors, (180,) *the product must cease to be a multiple of the number in which that prime is oftenest factor.*

184. Though the above rule is infallible, it will appear clumsy to one skilled in discovering the prime factors of numbers. For, by a mere inspection of the numbers 8, 9, 12, 15, 24, 27, we easily see that the only prime factors are 2, 3, and 5; that 8 is the highest power of 2; that 27 is the highest power of 3, and that 5 is only once factor: we, therefore, say, $8\times27\times5=1080$, the least common multiple.

185. The least common multiple of several numbers is *the greatest of those numbers; or, if not, it is a multiple of the greatest.* Wherefore we may often readily find the multiple thus: see *if the greatest number is a multiple of all the others;* if it is, *it is the least common multiple.* If it is not, try *twice that number.* If this does not succeed, *try* 3, 4, &c. *times the greatest.* Thus, if we would find the least common multiple of 6, 9, 18, 27, and 54, we see at once that it is 54. Again, if we would find the least common multiple of 2, 4, 6, 8, 12, 16, we first try if 16 is a common multiple of all the other numbers; it is not: we then try twice 16, which is 32; as this also fails, we try 3 times 16; and, as this succeeds, 48 is the required multiple.

Also, the least common multiple may often be conveniently found *by selecting the greatest numbers that are prime to each other, and finding their product.* Thus, if we would find the least common multiple of 2, 3, 4, 6, 8, 9, 12, 18, 24, and 36, we select the numbers 8 and 9, which are prime to each other; and, having multiplied them together, we find that their product 72 contains all the given numbers, and is, consequently, (179,) their least common multiple.

If the greatest number, or any other, be a multiple of any of the given numbers, *those numbers may, in operating by the general method, be omitted.* Thus, if we would find the least common multiple of 3, 4, 6, 9, 12, 18, 27, 48, and 54, as 54 is a common multiple of 3, 6, 9, 18, and 27, we omit these numbers. Again, as 48 is a common multiple of 4 and 12, we *omit these also.* We therefore seek the least common multiple of 48 and 54, and we find 432, which is also the least common multiple of all the given numbers.

Examples.

1. Required the least common multiple of 3, 5, 7, 4, and 11.
Answer 4620.

To multiply numbers with facility, much often depends on the order in which they are taken. Thus, in the present example, without writing a figure, we first say, 4 times 5 is 20; and, leaving 0 till the last, we say 3 times 2 is 6, and 7 times 6 is 42. Then, (126,) for 11 times 42, as the sum of 4 and 2 does not exceed 9, we place their sum 6 between those figures, and have 462. Lastly, we place 0 on the right, and have 4620 for the product.

2. Required the least common multiple of 2, 5, 10, 11, and 55.
Answer, 110.

3. Required the least common multiple of 5, 8, 9, 11, and 13.
Answer, 51480.

4. What is the least common multiple of the nine digits?
Answer, 2520.

5. What is the least common multiple of 3, 17, 18, 51, and 54?
Answer, 918.

6. What is the least common multiple of 6, 14, 39, 42, 78, and 91?
Answer, 546.

7. What is the least common multiple of 18, 28, 33, 42, 48, 63, and 99?
Answer, 11088.

8. What is the least common multiple of 6, 8, 12, 14, 16, 18, 21, and 22?
Answer, 11088.

186. The number which divides the greater of two numbers and the less, will (172) divide *their difference.* Supposing the difference greater than the less number, the number which divides the less and the difference will also divide *their difference.* Again, supposing the difference still greater than the less, the number which divides the less number and this second difference will also divide their difference. Now, this continual subtraction of the less number is the same as *the division of the greater number by the less.* The final remainder must be less than the less number, and we might reason, as above, with respect to this remainder and the less number. Again, if there be still a remainder, the same reasoning may be pursued with respect to this remainder and the first remainder. Pursuing the same method, as a greater number cannot divide a less, *the last remainder which succeeds in dividing the preceding will be*

the greatest common measure. Wherefore, to find the greatest common measure of two composite numbers, we have the following

Rule.

Divide the greater number by the less; if there is no remainder, the number divided by is the greatest common measure.

If there is a remainder, divide the less number by this remainder, and if this division gives no remainder, the number last divided by is the greatest common measure.

If there is still a remainder, divide the first remainder by the second; and thus continue, always dividing the preceding remainder by the last, till the division becomes exact.

The last divisor is *the greatest common measure.*

When the last divisor is a unit, *the numbers are prime to each other.*

Examples.

1. Required the greatest common measure of 9388 and 16429.

We proceed, according to the rule, thus:

```
9388) 16429 (1
       9388
      ------
       7041) 9388 (1
             7041
            ------
             2347) 7041 (3
                   7041
                   ----
                   ****
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and have 2347, the last divisor, for the required measure.

Because 2347 divides 7041, it will evidently divide 7041 + 2347, which is 9388. Again, because 2347 divides 7041 and 9388, it will divide their sum, which is 16429. Therefore 2347 is a common measure of 9388 and 16429.

No number greater than 2347 can be their common measure: for, if possible, let x be greater than 2347, and, at the same time, their common measure. Then, because x measures 9388 and 16429, it measures their difference 7041: also, because it measures 9388 and 7041, it measures their difference 2347; that is, a number greater than 2347 measures it, which is absurd. Therefore 2347 is the greatest common measure.

2. Required the greatest common measure of 3675 and 5880. Answer, 735.

3. Find the greatest common measure of 4437 and 5899. Answer, 17.

4. Find the greatest common measure of 23205 and 31395. Answer, 1365.

5. Required the greatest common measure of 46503 and 57546. Answer, 9.

6. Required the greatest common measure of 698 and 5343. Answer, —.

7. Required the greatest common measure of 34177 and 1012924. Answer, 11.

OF RATIO.

187. In division we say that the greater the dividend is, the greater will the quotient be, and the less the dividend, the less the quotient, (165.) This relation of the dividend and quotient is called *direct ratio.*

Again, the greater the divisor is, the less will the quotient be; and the less the divisor, the greater the quotient. This relation of the divisor and quotient is called *inverse ratio.*

188. A simple ratio is the relation which one number bears to another, with respect to *how often the one contains the other,* or to *what part the one is of the other;* and is, in either case, expressed *by that one divided by that other.* Thus, if we would have the ratio of 3 to 5, or find what part 3 is of 5: as 1 unit is $\frac{1}{5}$ of 5, it is plain that 3 units must be $\frac{3}{5}$; that is to say, the ratio of 3 to 5 is 3 divided by 5. Again, if we would have the ratio of 5 to 3, or find how often 5 contains 3, this is evidently $\frac{5}{3}$.

Let A and B be any two homogeneous quantities—that is, quantities, the units of which are alike; then, if A be greater than B, $\frac{A}{B}$ shows *how often* A *contains* B: and, if A be less than B, $\frac{A}{B}$ shows *what part* A *is of* B; for, B being a whole or unit divided into equal parts, determines the value of these parts, and A is the number of the same parts which constitute the fraction $\frac{A}{B}$. Wherefore, the ratio of A to B is $\frac{A}{B}$, and that of B to A is $\frac{B}{A}$. The ratio of 8 to 4 is $\frac{8}{4}=2$, and the ratio of 4 to 8 is $\frac{4}{8}=\frac{1}{2}$.

Hence, every fraction expresses a ratio; namely, *the ratio of the numerator to the denominator.*

The ratio $\frac{A}{B}$ is called *the reciprocal* of the ratio $\frac{B}{A}$; that is either ratio (or either fraction) is *the reciprocal of the other.*

189. A compound ratio is *that which involves two or more simple ratios.*

If two fractions have each the same denominator, the ratio of the one to the other is *the simple direct ratio of their numerators.* Thus, the ratio of $\frac{3}{7}$ to $\frac{5}{7}$ is the same as that of 3 to 5; namely, $\frac{3}{5}$; being (188) no more than the ratio of 3 units of a certain order to 5 units of the same order.

Again, if two fractions have each the same numerator, they have to one another *the inverse ratio of their denominators.* Thus, the ratio of $\frac{3}{5}$ to $\frac{3}{7}$ is $\frac{7}{5}$. For, the greater the number of parts in the unit, the less they are in value: therefore, whatever number of times 7 is greater than 5, so many times are the units of the fraction $\frac{3}{7}$ less than those of $\frac{3}{5}$; or, so many times are the units of $\frac{3}{5}$ greater than those of $\frac{3}{7}$. But $\frac{7}{5}$ is this number of times: therefore, $\frac{7}{5}$ is the ratio of $\frac{3}{5}$ to $\frac{3}{7}$. The ratio of $\frac{3}{5}$ to $\frac{3}{10}$ is $\frac{10}{5} = 2$; which shows that $\frac{3}{5}$ is twice $\frac{3}{10}$. Again, the ratio of $\frac{3}{10}$ to $\frac{3}{5}$ is $\frac{5}{10} = \frac{1}{2}$; which shows that $\frac{3}{10}$ is one-half of $\frac{3}{5}$.

Wherefore, any two fractions have to one another *the direct ratio of their numerators, and the inverse ratio of their denominators.* The ratio of one fraction to another, therefore, being compounded of both these ratios, is *a compound ratio.* Of this we shall treat more fully hereafter.

190. As a fraction is composed of one or more units, generated by the division of some standard unit into a certain number of equal parts, the number of which is signified by its denominator, we call one of these equal parts *the generic unit of the fraction.* Thus, the generic unit of $\frac{5}{8}$ is $\frac{1}{8}$, that of $\frac{5}{7}$ is $\frac{1}{7}$, &c. Now, as fractions, having like numerators, are to one another inversely as their denominators, the generic unit of one fraction is to that of another in this same ratio, which is found *by dividing the denominator of the second by that of the first.* Thus, the ratio of $\frac{1}{5}$ to $\frac{1}{6}$ is $\frac{6}{5}$; the ratio of $\frac{1}{7}$ to $\frac{1}{5}$ is $\frac{5}{7}$, &c. Hence we say, that fifths are to sixths as 6 to 5; fifths to ninths as 9 to 5, &c.

How many times greater are thirds than tenths?

191. A whole number is expressed in the form of a fraction *by giving it a unit for denominator;* this unit, being the generic unit of the number, both before and (190) after it is so expressed. Thus, $\frac{3}{1} = 3$. Now, as both terms of $\frac{3}{1}$ may be

multiplied by any number in the scale 2, 3, 4, 5, 6, &c., ad inf., without (165) altering the value of the number 3, it is evident that we can give this, or any other whole number, any denominator that we please, without at all altering its value—namely, *by first multiplying the whole number by that denominator.* Thus, if we would express 6 as a fraction, having 7 for denominator, we say $6 = \frac{6 \times 7}{7} = \frac{42}{7}$.

We have already defined a fraction to be *that which is less than a unit:* consequently, that which is equal to, or greater than, a unit, though in fractional form, is not really a fraction. Therefore, when the numerator is less than the denominator, we call the fraction *a proper fraction.* But, when the numerator is equal to, or greater than the denominator, we call the fraction *an improper fraction.* The expression $\frac{42}{7}$ is, therefore, an improper fraction.

Express each of the numbers 3, 4, 5, 6, 7 as an improper fraction, having 8 for denominator.

To reduce a mixed number to an improper fraction:

192. Express its integral part (191) as a fraction, having the same denominator as its fractional part: then, as the generic unit is the same in both parts, add their numerators together, and place the sum over the common denominator. Thus, if we have $4\frac{3}{5}$, we first put $4 = \frac{4 \times 5}{5} = \frac{20}{5}$, then $\frac{20}{5} + \frac{3}{5} = \frac{23}{5}$; that is, $4\frac{3}{5} = \frac{23}{5}$. In the same manner reduce the following numbers: $9\frac{2}{3}$, $11\frac{6}{7}$, $15\frac{4}{11}$, $14\frac{7}{12}$, and $34\frac{5}{13}$.

193. Having well understood the preceding article, we may proceed more expeditiously, thus: *multiply the integral part by the denominator of the fractional part; and, to the product, add the numerator of the fractional part, for a numerator, under which place the denominator of the fractional part.* Thus, $5\frac{1}{6} = \frac{5 \times 6 + 1}{6} = \frac{31}{6}$, and $13\frac{2}{5} = \frac{13 \times 5 + 2}{5} = \frac{67}{5}$.

So also the following:

1. $8\frac{8}{9} = \frac{80}{9}$; $25\frac{2}{7} = \frac{177}{7}$, and $106\frac{3}{10} = \frac{1063}{10}$.
2. $98\frac{13}{14} = \frac{1385}{14}$; $34\frac{19}{41} = \frac{1413}{41}$, and $504\frac{27}{40} = \frac{20187}{40}$.
3. $59\frac{13}{99} = \frac{5854}{99}$; $173\frac{17}{19} = \frac{3304}{19}$, and $429\frac{8}{31} = \frac{13307}{31}$.

194. A fraction is in its lowest terms, when those terms are prime to each other. Let $\frac{A}{B}$ be a fraction, of which the

terms A and B are prime to each other. Let D be less than B; and, if possible, let $\frac{A}{B}=\frac{C}{D}$, C being, as usual, a whole number. Then, if m be the value of the ratio $(\frac{D}{B})$ of a unit of A to a unit of C, $m\,A=C$. Now, $m\,A=\frac{DA}{B}$. But, B being greater than D, and prime to A, cannot (180) measure their product. Therefore, C cannot be a whole number. Thus, if we would reduce $\frac{3}{7}$ to *fifths:* as the ratio of $\frac{1}{7}$ to $\frac{1}{5}$ (190) is $\frac{5}{7}$, the ratio of $\frac{3}{7}$ to $\frac{1}{5}$ is 3 times $\frac{5}{7}$ or $\frac{15}{7}=2\frac{1}{7}$; that is, $\frac{3}{7}\div\frac{1}{5}=2\frac{1}{7}$. Hence, (141,) $2\frac{1}{7}\times\frac{1}{5}=\dfrac{2\frac{1}{7}}{5}$ is the value of $\frac{3}{7}$ expressed in fifths. Proof, $\dfrac{2\frac{1}{7}}{5}\times\frac{7}{7}=\frac{15}{35}=\frac{3}{7}$. Hence, it is evident that, to reduce the fraction $\frac{A}{B}$ to an equivalent fraction, having another number, D, for its denominator, and C a whole number, D must not only be greater than B, but, as A is prime, D must be *a multiple of* B.

To reduce a fraction to its lowest terms:

195. *Divide both terms by their greatest common measure.* The quotients, having no longer any common factor, are prime to each other, and are, consequently, (194,) the lowest terms of the fraction. Thus, if we require the lowest terms of $\frac{6209}{11513}$, we first find (186) that the greatest common measure of 6209 and 11513 is 887. We then divide both terms of $\frac{6209}{11513}$ by 887; which (165) has no effect upon its value, and gives $\frac{7}{13}$ for its lowest terms.

Examples.

1. $\frac{195}{455}=\frac{3}{7}$; $\frac{580}{928}=\frac{5}{8}$; $\frac{714}{1785}=\frac{2}{5}$ and $\frac{108}{684}=\frac{3}{19}$.
2. $\frac{3675}{5880}=\frac{5}{8}$; $\frac{4437}{5899}=\frac{261}{347}$ and $\frac{56887}{69344}=\frac{5167}{6304}$.
3. $\frac{12989}{13827}=\frac{31}{33}$; $\frac{13549}{23113}=\frac{17}{29}$, and $\frac{698}{5343}=$ ———.
4. $\frac{14697}{83766}=\frac{213}{1214}$; and $\frac{73485}{235152}=\frac{5}{16}$.

196. The lowest terms may often be obtained with greater facility, *by dividing the terms of the given fraction by whatever number we can discover to be a common measure;* (175 to 178; also 173;) and, if not the greatest, continuing so to divide, *until the terms are prime to each other.* For example, $\frac{504}{1344}=\frac{63}{168}=\frac{3}{8}$. First, as the unit figure in each is 4, and the tens even, we see that 2 is more than once factor in each term; we therefore try 8, and have $\frac{63}{168}$ for the result. Then,

by adding the figures of each term, we find that 3 is factor. Also, as the tens in each are double of the unit figure, we know that 7 (or indeed 21) is factor; wherefore, dividing by 21, we have $\frac{3}{8}$.

Or, having found $\frac{63}{168}$, we may observe that the unit figures, 3 and 8, are prime to each other, and that the tens, 6 and 16, are in the same ratio; and hence that the fraction is $\frac{3}{8}$. In the same manner we easily discover that $\frac{126}{147} = \frac{6}{7}$; $\frac{155}{279} = \frac{5}{9}$; $\frac{968}{1089} = \frac{8}{9}$, and $\frac{753}{1004} = \frac{3}{4}$.

197. When the terms are nearly equal, *try to divide both by their difference*, (173.) *If this does not succeed, try its half, third, fourth, &c.* To reduce $\frac{12989}{13827}$, we take the difference between the terms, which is 838. But, because no whole number of times the unit figure 8 can produce 9 or 7, we take half the difference, which is 419. Dividing both terms by this, we have $\frac{31}{33}$ for the terms sought. Thus, we find that

$$\frac{30919}{32812} = \frac{49}{52}; \text{ and } \frac{106097}{111469} = \frac{79}{83}.$$

198. In order to add two or more numbers together, or subtract one from another, *the numbers must be homogeneous:* that is, *the generic unit in each must be the same.* For, if we would add $\frac{3}{4}$ and $\frac{3}{7}$, the units of which are not alike, we cannot combine the numerators by saying 3 and 3 is 6; because this number could neither be denominated *fourths* nor *sevenths*, although comprising no other denomination.

Again, if we would subtract $\frac{3}{7}$ from $\frac{3}{4}$, we cannot say 3 from 3, *nought;* because this would imply that there is no difference between the fractions, which (189) is absurd.

199. It is obvious, then, that before we can add dissimilar fractions together, or subtract one from the other, their units must be rendered similar: that is, (190,) *they must all have the same denominator.* This is called reducing them to *a common denominator.* But (194) this must be a multiple of each denominator; and, for convenience in calculation, the less it is, the better. Therefore, we take for a com. denom. *the least common multiple of all the denominators.*

To reduce Dissimilar Fractions to their Least Common Denominator.

200. First find the least common multiple of all the denominators, and call it *the common denominator.*

2. Multiply each numerator by the number of times that its denominator is contained in the common denominator.

3. Place each of the several products over the common denominator, which will give the required fractions.

Examples.

1. Reduce $\frac{3}{4}$ and $\frac{3}{7}$ to a common denominator.

When there are two fractions with prime denominators, each denominator shows how often the other is contained in the product of the two denominators; which is their least common multiple. Wherefore, by the Rule: *multiply the numerator of each fraction by the denominator of the other, and place the product over the com. denom.* This is the same as to multiply the two terms of each fraction by the denom. of the other, which (165) does not alter its value. Thus, we have $3\times7=21$; $3\times4=12$, and $7\times4=28$. Therefore,

$\frac{21}{28}$ and $\frac{12}{28}$ are the fractions required.

Or thus:

$$\frac{3}{4}=\frac{3\times7}{4\times7}=\frac{21}{28};\text{ and }\frac{3}{7}=\frac{3\times4}{7\times4}=\frac{12}{28}.$$

When there are more than two fractions having prime denominators.—The product of the denominators being, as before, the least com. denom., *multiply each numerator into all the denominators except its own, and place the product over the com. denom.* It is easy to see, as above, that this is the same as to multiply both terms of each fraction by the product of the denominators of all the others.

2. Reduce $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{7}$ to a common denominator.

$$\left.\begin{array}{l}2\times5\times7=70\\3\times3\times7=63\\5\times5\times3=75\end{array}\right\}\text{ numerators}\qquad 3\times5\times7=105\text{ com. denom.}$$

$\frac{70}{105}$, $\frac{63}{105}$, and $\frac{75}{105}$, required fractions.

3. $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{3}{5}$, are equivalent to $\frac{15}{60}$, $\frac{40}{60}$, and $\frac{36}{60}$.
4. $\frac{4}{5}$, $\frac{4}{9}$, $\frac{6}{7}$, " " " $\frac{252}{315}$, $\frac{140}{315}$, $\frac{270}{315}$.
5. $\frac{3}{4}$, $\frac{5}{9}$, $\frac{6}{7}$, $\frac{7}{11}$ " " " $\frac{2079}{2772}$, $\frac{1540}{2772}$, $\frac{2376}{2772}$, $\frac{1764}{2772}$.
6. $\frac{3}{8}$, $\frac{5}{13}$, $\frac{8}{15}$, " " " $\frac{585}{1560}$, $\frac{600}{1560}$, $\frac{832}{1560}$.
7. $\frac{7}{19}$, $\frac{9}{10}$, $\frac{13}{23}$, " " " $\frac{1610}{4370}$, $\frac{3933}{4370}$, $\frac{2470}{4370}$.

201. When the denominators are not prime to each other, we find their least com. mult., (183 or 185,) which we take for a com. denom. Then, having placed the fractions by the side of each other, and drawn a line underneath, we divide the com. denom. by

the denominator of each fraction, placing the quotient below the line under the denominator which gave it. Lastly, we multiply the two terms of each fraction by the corresponding number below the line, and have the fractions sought.

We need not multiply each denominator by the number below the line, because we already know the result. Also, when the scholar becomes skilful, he may omit the placing of the numbers below the line.

To reduce $\frac{2}{3}$, $\frac{3}{4}$, $\frac{7}{8}$, and $\frac{5}{16}$ to a com. denom., we place the numbers thus: $\begin{array}{cccc} \frac{2}{3} & \frac{3}{4} & \frac{7}{8} & \frac{5}{16} \\ \hline 16 & 12 & 6 & 3 \end{array}$ and having found (185) that the least com. denom. is 48, we divide this successively by 3, 4, 8, and 16, and place the quotients 16, 12, 6, and 3, each under the number which gave it. Lastly, we multiply the two terms of each fraction, or rather, the numerator only, by the number below the line, and have $\frac{32}{48}$, $\frac{36}{48}$, $\frac{42}{48}$, and $\frac{15}{48}$ for the required fractions.

We easily see (190) that each of the numbers 16, 12, 6, and 3, respectively, shows *the number of times that the new common generic unit* $\frac{1}{48}$ *is less than that of the fraction under which the number stands;* and that in multiplying the numerator by this number, we have a number of parts in each new fraction, which is *just as many times greater as the number of times that the parts are less in value.* Wherefore, *the value of the fraction is not altered.*

Examples.

1. $\frac{5}{6}$, $\frac{11}{12}$, and $\frac{7}{15}$, are equal to $\frac{50}{60}$, $\frac{55}{60}$, and $\frac{28}{60}$.
2. $\frac{5}{8}$, $\frac{7}{9}$, $\frac{35}{36}$, $\frac{11}{24}$, " " $\frac{45}{72}$, $\frac{56}{72}$, $\frac{70}{72}$, $\frac{33}{72}$.
3. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{7}{12}$, " " $\frac{16}{24}$, $\frac{18}{24}$, $\frac{4}{24}$, $\frac{3}{24}$, $\frac{14}{24}$.
4. $\frac{5}{6}$, $\frac{8}{9}$, $\frac{17}{18}$, $\frac{26}{27}$, $\frac{37}{54}$, " " $\frac{45}{54}$, $\frac{48}{54}$, $\frac{51}{54}$, $\frac{52}{54}$, $\frac{37}{54}$.
5. $\frac{2}{3}$, $\frac{16}{17}$, $\frac{19}{24}$, $\frac{31}{51}$ " " $\frac{272}{408}$, $\frac{384}{408}$, $\frac{323}{408}$, $\frac{248}{408}$.

SECTION IX.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF VULGAR FRACTIONS AND MIXED NUMBERS;—ALSO DIVISION BY FACTORS.

Addition of Vulgar Fractions.

202. WHEN the fractions to be added have a com. denom., *we add all the numerators, and place the sum over the com.*

denom. The fraction thus formed, whether proper or improper, is the sum of the given fractions.

The numerators, that is to say, the fractions, being each composed of units of the order signified by the com. denom., their sum is a number of units of the same order; and, for this reason, we give it the same denominator

Thus, $\frac{3}{7}+\frac{2}{7}=\frac{5}{7}$.

But when they have not a common denominator, (198,) *we first reduce them to a com. denom.*, (200 or 201,) *and proceed as before.* Thus, to find the sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{7}$, we first reduce these fractions to a com. denom., (200,) and have for their equivalents $\frac{21}{42}$, $\frac{28}{42}$, and $\frac{18}{42}$. Then to find their sum we say, $\frac{21}{42}+\frac{28}{42}+\frac{18}{42}=\frac{21+28+18}{42}=\frac{67}{42}=1\frac{25}{42}$. Wherefore, the sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{7}$ is $1\frac{25}{42}$.

Examples.

1. $\frac{3}{4}+\frac{5}{9}+\frac{6}{7}=\frac{189}{252}+\frac{140}{252}+\frac{216}{252}=\frac{189+140+216}{252}=\frac{545}{252}=2\frac{41}{252}$;

Hence, the sum of $\frac{3}{4}$, $\frac{5}{9}$, and $\frac{6}{7}$ is $2\frac{41}{252}$. Let the expressions in the following examples be written in the same manner:

2. $\frac{3}{5}+\frac{4}{7}+\frac{5}{11}=\frac{a}{d}+\frac{b}{d}+\frac{c}{d}=\frac{a+b+c}{d}=\frac{s}{d}=\frac{626}{385}=1\frac{241}{385}$.

The letters *a b c* represent the new numerators, *s* their sum, and *d* the common denominator.

3. $\frac{2}{3}+\frac{5}{6}+\frac{7}{8}=\frac{16}{24}+\frac{20}{24}+\frac{21}{24}=\frac{16+20+21}{24}=\frac{57}{24}=\frac{19}{8}=2\frac{3}{8}$.
4. $\frac{1}{6}+\frac{6}{7}+\frac{13}{14}+\frac{1}{21}=\frac{a}{d}+\frac{b}{d}+\frac{c}{d}+\frac{e}{d}=\frac{a+b+c+e}{d}=\frac{s}{d}=2$.
5. $\frac{1}{9}+\frac{2}{7}+\frac{5}{21}+\frac{16}{63}=\frac{8}{9}$; also, $\frac{1}{11}+\frac{2}{5}+\frac{28}{55}=1$.
6. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{7}{8}+\frac{8}{9}=5\frac{113}{360}$.
7. $\frac{10}{11}+\frac{5}{12}+\frac{9}{13}+\frac{16}{17}=2\frac{27983}{29172}$.
8. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}=1\frac{2089}{2520}$.

Addition of Mixed Numbers.

203. Mixed numbers may be added either by first changing them to improper fractions, and then proceeding as above, or by first finding the sum of the integral parts; then, that of the fractional parts; and lastly, the sum of the two sums.

Examples.

1. To add $5\frac{1}{2}$, $6\frac{3}{4}$, and $7\frac{5}{8}$, we may proceed thus: $5\frac{1}{2} = \frac{11}{2}$; $6\frac{3}{4} = \frac{27}{4}$; $7\frac{5}{8} = \frac{61}{8}$; (193,) therefore, $5\frac{1}{2} + 6\frac{3}{4} + 7\frac{5}{8} = \frac{11}{2} + \frac{27}{4} + \frac{61}{8} = \frac{44}{8} + \frac{54}{8} + \frac{61}{8} = \frac{44+54+61}{8} = \frac{159}{8} = 19\frac{7}{8}$. Or thus: $5 + 6 + 7 = 18$; the sum of the integral parts: then, $\frac{1}{2} + \frac{3}{4} + \frac{5}{8} = \frac{4}{8} + \frac{6}{8} + \frac{5}{8} = \frac{4+6+5}{8} = \frac{15}{8} = 1\frac{7}{8}$; the sum of the fractional parts; and lastly, $18 + 1\frac{7}{8} = 19\frac{7}{8}$, as before.

The student may pursue the method which seems most convenient, according to circumstances. The one also proves the other.

2. $2\frac{1}{2} + 1\frac{1}{9} + 3\frac{6}{7} + \frac{67}{126} = 8$.
3. $101\frac{1}{6} + 315\frac{2}{7} + \frac{19}{28} = 417\frac{11}{84}$.
4. $4\frac{3}{4} + 5\frac{2}{3} + 1\frac{5}{9} + 12\frac{1}{5} = 24\frac{31}{180}$.
5. $\frac{1}{8} + \frac{3}{11} + 8\frac{1}{3} + 19\frac{29}{40} + 62\frac{7}{9} = 91\frac{463}{1980}$.

Subtraction of Fractions.

204. When the given fractions have a com. denom., we subtract the numerator of the less from that of the greater, and place the remainder over the com. denom.; thus, $\frac{6}{7} - \frac{2}{7} = \frac{4}{7}$.

The numerators being each composed of units of the order signified by the com. denom., their difference is evidently a number of units of the same order; for which reason we give it the same denominator.

When the fractions have different denominators, *we reduce them to a com. denom.*, as in addition; because we cannot subtract units of different orders; after which we proceed as before.

Examples.

1. $\frac{6}{7} - \frac{2}{3} = \frac{18}{21} - \frac{14}{21} = \frac{4}{21}$.
2. $\frac{5}{9} - \frac{1}{5} = \frac{16}{45}$; $\frac{17}{19} - \frac{8}{9} = \frac{1}{171}$; $\frac{5}{6} - \frac{3}{8} = \frac{11}{24}$.
3. $\frac{7}{8} - \frac{4}{5} = \frac{3}{40}$; $\frac{11}{12} - \frac{5}{9} = \frac{13}{36}$; $\frac{16}{51} - \frac{3}{17} = \frac{7}{51}$.
4. $\frac{5}{13} - \frac{8}{61} = \frac{201}{793}$; $\frac{103}{140} - \frac{1}{2} = \frac{33}{140}$; $\frac{189}{190} - \frac{9}{11} = \frac{369}{2090}$.
5. $\frac{83}{84} - \frac{7}{8} = \frac{19}{168}$; $\frac{95}{96} - \frac{35}{36} = \frac{5}{288}$; $\frac{255}{765} - \frac{153}{459} =$ ——
6. $\frac{32}{144} - \frac{34}{153} =$ ——; $\frac{213}{497} - \frac{207}{483} =$ ——; $\frac{42}{154} - \frac{44}{165} = \frac{1}{165}$.

Subtraction of Mixed Numbers.

205. We may reduce the mixed numbers to improper fractions; and, when the denominators are not alike, bring them to a common denominator, and subtract as above. For example: if, from $7\frac{1}{8}$, we would subtract $2\frac{3}{7}$, we say $7\frac{1}{8} = \frac{57}{8}$, and $2\frac{3}{7} = \frac{17}{7}$; therefore,

$$7\tfrac{1}{8} - 2\tfrac{3}{7} = \frac{57}{8} - \frac{17}{7} = \frac{399}{56} - \frac{136}{56} = \frac{399-136}{56} = \frac{263}{56} = 4\tfrac{39}{56}.$$

But, when the integral part of one or both numbers is very great, the following method will be found more convenient: If the fractional parts have different denominators, we first reduce them to a com. denom. We then write the less mixed number under the greater, and, having drawn a line underneath, we subtract the numerator of the lower fraction from that of the upper one, and place the remainder over the com. denom. for the difference of the fractions. Lastly, we find the difference of the whole numbers, as usual.

When the numerator of the lower fraction is the greater of the two, we subtract it from the sum of the terms of the upper fraction; and, having placed the remainder over the com. denom., we carry one to the unit figure of the lower number, and proceed as usual.

As the denominator is a unit, and the numerator the fraction, when we add the denominator to the numerator, *we add a unit to the fraction*; therefore, having added a unit to the upper number, we add a unit to the lower, by which means both numbers are increased alike, which cannot affect the difference.

Examples.

1. To subtract $2\frac{3}{7}$ from $7\frac{1}{8}$, we first find that $\frac{1}{8}$ and $\frac{3}{7}$ are equivalent to $\frac{7}{56}$ and $\frac{24}{56}$; we then place the numbers thus:

$$\begin{array}{r} 7\frac{7}{56} \\ \underline{2\frac{24}{56}} \\ 4\frac{39}{56} \end{array}$$

and, as 24 is greater than 7, we say $56 - 24 = 32$; and $32 + 7 = 39$, which we place over 56. Then, having added a unit to the upper number, we add a unit to the lower number 2, saying, 3 from 7, *four.* Wherefore, $7\frac{1}{8} - 2\frac{3}{7} = 4\frac{39}{56}$.

2. $101\frac{13}{15} - 94\frac{11}{13} = \frac{1528}{15} - \frac{1233}{13} = \frac{a}{d} - \frac{b}{d} = \frac{a-b}{d} = \frac{c}{d} = 7\frac{4}{195}$.

Or thus : $\frac{13}{15}$ and $\frac{11}{13}$ are equivalent to $\frac{169}{195}$ and $\frac{165}{195}$; then,

$$\begin{array}{lr} \text{from} & 101\frac{169}{195} \\ \text{we take} & 94\frac{165}{195} \\ \hline \text{and have} & 7\frac{4}{195} \end{array} \text{ for remainder, as before.}$$

Here, as $\frac{13}{15}$ is greater than $\frac{11}{13}$, we do not add a unit to either number, as in the preceding example.

3. To subtract $94\frac{11}{13}$ from 101, we place the numbers thus:

$$\begin{array}{r} 101 \\ 94\frac{11}{13} \\ \hline 6\frac{2}{13} \end{array}$$

and, as there is nothing above $\frac{11}{13}$, we subtract it from a unit, and carry a unit to the figure 4, as in the first example. Now, in reducing a unit to a fraction, having 13 for denominator, (191,) we have 13 also for the numerator; but, as this is the same as the denominator, we subtract the numerator of $\frac{11}{13}$ from its own denominator, and, placing the difference 2 over 13, we have $\frac{2}{13}$ for the difference between $\frac{11}{13}$ and a unit; we then carry a unit to 4, and, having subtracted, we find that $6\frac{2}{13}$ is the difference between the given numbers.

The difference between a fraction and a unit is called *the complement of that fraction* : thus, $\frac{11}{13}$ and $\frac{2}{13}$ are the complements of each other.

4. $15\frac{3}{4} - 3\frac{7}{8} = 11\frac{7}{8}$; and $15\frac{3}{4} - 11\frac{7}{8} = 3\frac{7}{8}$.

5. $9\frac{31}{40} - 6\frac{7}{8} = 2\frac{9}{10}$; and $9\frac{31}{40} - 2\frac{9}{10} = 6\frac{7}{8}$.

6. $28\frac{2}{3} - 9\frac{11}{12} = 18\frac{3}{4}$; and $28\frac{2}{3} - 18\frac{3}{4} = 9\frac{11}{12}$.

7. $65\frac{5}{17} - 19\frac{11}{12} = 45\frac{77}{204}$; and $9 - 4\frac{3}{7} = 4\frac{4}{7}$.

8. $101 - 6\frac{2}{13} = 94\frac{11}{13}$; and $15\frac{1}{7} - 14\frac{16}{21} = \frac{8}{21}$.

9. $1043\frac{7}{19} - 65\frac{5}{17} = 978\frac{24}{323}$. Prove this and the succeeding example by addition and subtraction.

10. $3171\frac{1}{17} = 980\frac{7}{51} = 2190\frac{47}{51}$.

Multiplication of Fractions.

206. To find the product of several fractions, *we multiply all the numerators together for a numerator, and all the denominators for a denominator.* The fraction thus formed is *the product.*

For example, to multiply $\frac{3}{4}$ by $\frac{2}{3}$. If we (165) first multiply by 2, we have $\frac{6}{4}$. But $\frac{2}{3}$ is (145) the third part of 2; hence, the product $\frac{6}{4}$ is 3 times too great, and must be divided by 3. Now, this is done (165) in multiplying the denominator 4 by 3, which gives $\frac{6}{12}$ for the quotient; therefore,

$$\frac{3}{4}\times\frac{2}{3}=\frac{3\times 2}{4\times 3}=\frac{6}{12}=\tfrac{1}{2}$$

Or, to multiply $\frac{3}{4}$ by 2; we divide (165) its denominator by 2, and have $\frac{3}{2}$ for the product. But this, as above, is just 3 times the true product: we, therefore, (165,) divide its numerator by 3, and have $\frac{1}{2}$ for the true product, as before.

Operation.

$$\frac{\cancel{3}}{\cancel{4}}\times\frac{\cancel{2}}{\cancel{3}}=\tfrac{1}{2}$$

The striking out of a numerator and denominator, as here shown, is called *cancelling*. We do not always write the quotient 1 above or below the figure wholly cancelled; because we know that, when all the numerators are cancelled, the result is 1; seeing that any power of 1 is 1.

The reasoning applied above to the multiplication of two fractions, will apply to the multiplication of their product and a third fraction; to that product and a fourth, &c., and, consequently, to the continued multiplication of any number of fractions. Hence, we see that, in multiplying any number of fractions, the resulting numerator is *the product of all the numerators*, and the denominator, *the product of all the denominators.*

207. Let us observe that, in multiplying by a fraction, we not only perform a multiplication, but also a division; and, as the number by which we divide is greater than that by which we multiply, the quantity multiplied by a fraction, instead of being increased, *is diminished.*

Examples.

1. To multiply $\frac{1}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$ together, we proceed thus:

$$\frac{1\times\cancel{3}\times\cancel{4}\times\cancel{5}}{\cancel{3}\times\cancel{4}\times\cancel{5}\times 6}=\tfrac{1}{6}$$

The lines drawn through the numbers 3, 4, 5, show that they cancel each other; that is to say, because the value of $\frac{1}{6}$ would not be altered, (165,) if both its terms were multiplied by the product 3×4×5, we omit the multiplication of these numbers, which saves the trouble, not only of this multiplication, but also of reducing the final product to its lowest terms.

When two opposite terms are equal, *they may be omitted:*

when one is a multiple of the other, *we may divide the greater by the less, and do with the quotient as we should have done with the greater:* also, when they have a common measure, *divide both by it, and operate with the quotients as with the numbers;* because, in each case, the effect is the same as to divide both terms of the final product by the same number, which (165) does not alter its value.

$$2.\quad \tfrac{9}{13}\times\tfrac{5}{24}\times\tfrac{26}{27}\times\tfrac{36}{55}=\tfrac{5}{55}=\tfrac{1}{11}.$$

The prime factors of $9\times26\times36$ and of $13\times24\times27$ are the same, namely, 3^4, 2^3, and 13: therefore, omitting these numbers, we have $\tfrac{5}{55}=\tfrac{1}{11}$ for the total product.

3. $\tfrac{1}{2}\times\tfrac{5}{6}\times\tfrac{7}{8}=\tfrac{35}{96}$, and $\tfrac{3}{4}\times\tfrac{8}{9}\times\tfrac{15}{16}=\tfrac{5}{8}$.
4. $\tfrac{3}{7}\times\tfrac{5}{9}\times\tfrac{21}{25}=\tfrac{1}{5}$, and $\tfrac{4}{9}\times\tfrac{14}{17}\times\tfrac{51}{56}=\tfrac{1}{3}$.
5. $\tfrac{2}{7}\times\tfrac{6}{11}\times\tfrac{11}{12}=\tfrac{1}{7}$, and $\tfrac{18}{19}\times\tfrac{2}{9}\times\tfrac{57}{58}=\tfrac{6}{29}$.
6. $\tfrac{3}{7}\times\tfrac{2}{5}\times\tfrac{8}{9}\times\tfrac{7}{8}\times\tfrac{15}{16}=\tfrac{1}{8}$, and $\tfrac{5}{13}\times\tfrac{16}{17}\times\tfrac{21}{23}=\tfrac{1680}{5083}$.
7. $\tfrac{2}{3}\times\tfrac{3}{7}\times\tfrac{5}{8}\times\tfrac{9}{10}\times\tfrac{11}{16}\times\tfrac{41}{44}=\tfrac{369}{3584}$.

208. To multiply a fraction by a whole number, is the same as to multiply the whole number by the fraction; for, in either case, (191,) having expressed the whole number as a fraction, we multiply the whole number and the numerator of the fraction together, and divide the product by the denominator, (206.) Thus:

$$2136\times\tfrac{3}{4}=\tfrac{2136}{1}\times\tfrac{3}{4}=\tfrac{6408}{4}=1602.$$

Or, by cancelling, if practicable, first divide the whole number by the denominator of the fraction, and multiply the numerator by the quotient. Or, if the whole number divide the denominator of the fraction, divide the numerator by the quotient. Thus:

$$2136\times\tfrac{3}{4}=534\times3=1602.$$

Also, when the numerator and denominator differ, as in the present case, by only one unit, we may, when we have divided the whole number by the denominator, subtract the quotient from the whole number: thus, having found the quotient 534, we say, $2136-534=1602$, as before.

Therefore, to take $\tfrac{5}{6}$, $\tfrac{6}{7}$, $\tfrac{7}{8}$, $\tfrac{8}{9}$, &c. of a number, is *to diminish it by* $\tfrac{1}{6}$, $\tfrac{1}{7}$, $\tfrac{1}{8}$, $\tfrac{1}{9}$, *&c. of itself.*

When the whole number is the same as the denominator of the fraction, the result is *the numerator.* Thus,

$$\tfrac{3}{4}\times4=3;\quad \tfrac{11}{12}\times12=11, \text{ \&c.}$$

Examples.

1. $\frac{5}{6}\times3=\frac{5}{6}=\frac{3}{1}=\frac{15}{6}=\frac{5}{2}=2\frac{1}{2}$. Or, $\frac{5}{\cancel{6}}\times\cancel{3}=\frac{5}{2}=2\frac{1}{2}$.
2. $\frac{7}{8}\times4=3\frac{1}{2}$; $\frac{11}{12}\times6=5\frac{1}{2}$; $\frac{17}{18}\times3=2\frac{5}{6}$; $\frac{p}{q}\times q=p$.
3. $\frac{7}{8}\times11=9\frac{5}{8}$; $134\times\frac{12}{13}=123\frac{9}{13}$; $\frac{17}{19}\times204=182\frac{10}{19}$.

Multiplication of Mixed Numbers.

209. Reduce the mixed numbers to improper fractions, and multiply them (206) as fractions. Also, when there are whole numbers, express them as fractions; and, in all cases, cancel as much as possible.

Examples.

1. $53\times5\frac{3}{8}=\frac{53}{1}\times\frac{43}{8}=\frac{2279}{8}=284\frac{7}{8}$. Or (138) thus: $53\times5=265$, and $53\times\frac{3}{8}=\frac{159}{8}=19\frac{7}{8}$: then, $265+19\frac{7}{8}=284\frac{7}{8}$.

2. $2\frac{1}{5}\times4\frac{4}{5}\times3\frac{1}{8}=\frac{11}{\cancel{5}}\times\frac{\overset{3}{\cancel{24}}}{\cancel{5}}\times\frac{\cancel{25}}{\cancel{8}}=33$.

3. $15\frac{2}{3}\times5\frac{7}{8}$. (138.) $15\frac{2}{3}$

			$5\frac{7}{8}$
Product of	15×5	-	75
"	$\frac{2}{3}\times5$	-	$3\frac{1}{3}$ or $\frac{8}{24}$
"	$15\times\frac{7}{8}$	-	$13\frac{1}{8}$ or $\frac{3}{24}$
"	$\frac{2}{3}\times\frac{7}{8}$	-	- $\frac{14}{24}$
			$92\frac{1}{24}$

By the general method, $15\frac{2}{3}\times5\frac{7}{8}=\frac{47}{3}\times\frac{47}{8}\times\frac{2209}{24}=92\frac{1}{24}$, as before.

4. $\frac{9}{13}\times2\frac{8}{9}\times\frac{1}{2}=1$; $2\frac{3}{5}\times3\frac{3}{4}\times4=39$; $33\frac{1}{3}\times10\frac{4}{5}=360$.
5. $\frac{2}{3}\times3\frac{7}{11}\times2\frac{3}{4}\times\frac{13}{14}=6\frac{4}{21}$; $6\frac{1}{2}\times3\frac{1}{26}\times2=39\frac{1}{2}$.
6. $94\frac{6}{7}\times77\frac{17}{43}=7341\frac{151}{301}$; $69\frac{4}{19}\times39\frac{3}{25}=2707\frac{49}{95}$.

210. The calculation is often facilitated *by striking a factor from one number and introducing it into another.* Thus, striking the factor 11 from $33\frac{11}{12}$, we have $3\frac{1}{12}$; then introducing it into $13\frac{9}{11}$, we have 152, and multiplying the results, instead of the numbers, we have $468\frac{2}{3}$ for the product, as follows:

1. $13\frac{9}{11} \times 33\frac{11}{12} = 152 \times 3\frac{1}{12} = 456 + 12\frac{2}{3} = 468\frac{2}{3}$.
2. $6\frac{1}{4} \times 16\frac{8}{9} = 50 \times 2\frac{1}{9} = 100 + 5\frac{5}{9} = 105\frac{5}{9}$.
3. $16\frac{2}{3} \times 18\frac{3}{4} = 50 \times 6\frac{1}{4} = 312\frac{1}{2}$.
4. $18\frac{3}{7} \times 14\frac{7}{9} = 129 \times 2\frac{1}{9} = 258 + 14\frac{1}{3} = 272\frac{1}{3}$.
5. $115\frac{3}{5} \times 250\frac{5}{6} = 578 \times 50\frac{1}{6} = 28900 + 96\frac{1}{3} = 28996\frac{1}{3}$.

211. A fraction of a fraction is called *a compound fraction.* Thus $\frac{3}{5}$ of $\frac{7}{8}$ is a compound fraction.

To take $\frac{1}{5}$ of $\frac{7}{8}$ is (144) to divide $\frac{7}{8}$ by 5; that is, (165,) to multiply its denominator 8 by 5, which gives $\frac{7}{40}$; therefore, $\frac{3}{5}$ of $\frac{7}{8}$ must be 3 times $\frac{7}{40}$; and this (165) is found in multiplying the numerator 7 by 3, which gives $\frac{21}{40}$. Hence, *we may substitute the sign* × *for the word of,* and proceed as in multiplication of fractions. Wherefore, $\frac{3}{5}$ of $\frac{7}{8} = \frac{3}{5} \times \frac{7}{8} = \frac{21}{40}$. Also, it is evident that $\frac{3}{5}$ of $\frac{7}{8}$ of $\frac{9}{11} = \frac{21}{40}$ of $\frac{9}{11}$; that is, as above, $\frac{21}{40} \times \frac{9}{11} = \frac{189}{440}$. We see, therefore, that to reduce a compound fraction to a simple one, *the fractions composing the compound fraction must all be multiplied together.*

Examples.

1. $\frac{1}{8}$ of $\frac{2}{3}$ of $9 = \frac{1}{8} \times \frac{2}{3} \times \frac{9}{1} = \frac{3}{4}$; $\frac{1}{5}$ of $\frac{6}{7}$ of $\frac{11}{13} = \frac{66}{455}$.
2. $\frac{1}{2}$ of $\frac{5}{6}$ of $\frac{7}{8} = \frac{35}{96}$; $\frac{2}{3}$ of $\frac{9}{10}$ of $25 = 15$.
3. $\frac{1}{2}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7} = \frac{2}{7}$; $\frac{4}{5}$ of $\frac{5}{8}$ of $32 = 16$.
4. $\frac{7}{8}$ of $\frac{16}{17}$ of $51\frac{2}{3} = 42\frac{28}{51}$; $\frac{1}{11}$ of $\frac{9}{23}$ of $28\frac{1}{9} = 1$.

Division of Fractions.

212. To divide a quantity Q by any fraction $\frac{n}{d}$, *multiply the dividend by the reciprocal of the divisor;* that is, by $\frac{d}{n}$. Thus,

$$Q \div \frac{n}{d} = Q \times \frac{d}{n} = \frac{Qd}{n}, \text{ the required quotient.}$$

First, $\frac{Q}{n}$ is the quantity divided by the numerator. But (208) $n = \frac{n}{d} \times d$: that is, n is d times too great; consequently, (165,) $\frac{Q}{n}$ is d times less than it should be; wherefore, (165,) Q must be multiplied by d, which gives $\frac{Qd}{n}$ for the true quotient.

Examples.

1. $24 \div \frac{3}{8} = 24 \times \frac{8}{3} = 8 \times 8 = 64$, and $24 \div \frac{3}{4} = 24 \times \frac{4}{3} = 32$.
2. $36 \div \frac{7}{8} = \frac{36 \times 8}{7} = \frac{288}{7} \times 41\frac{1}{7}$, and $36 \div \frac{7}{9} = 46\frac{2}{7}$.

3. $43 \div \frac{3}{11} = 157\frac{2}{3}$, and $99 \div \frac{1}{25} = 2475$.

4. $528 \div \frac{11}{12} = 576$, and $528 \div \frac{11}{24} = 1152$.

213. Let Q be any fraction $\frac{p}{q}$. Then, $\frac{p}{q} \div \frac{n}{d} = \frac{p}{q} \times \frac{d}{n} = \frac{pd}{nq}$. For $\frac{p}{q} \div n$ is $\frac{p}{nq}$, (165;) but, as n is d times too great, $\frac{p}{nq}$ is d times less than it should be; wherefore, we multiply p by d, and have $\frac{pd}{nq}$ for the true quotient.

To divide by a fraction, therefore, we have this simple rule: *invert it,* (that is, turn it upside down,) *and multiply.*

Examples.

1. $\frac{5}{16} \div \frac{7}{8} = \frac{5}{16} \times \frac{8}{7} = \frac{5}{14}$; $\frac{11}{12} \div \frac{11}{24} = \frac{11}{12} \times \frac{24}{11} = 2$.
2. $\frac{3}{10} \div \frac{4}{5} = \frac{3}{8}$; $\frac{5}{6} \div \frac{7}{10} = 1\frac{4}{21}$; $\frac{5}{6} \div \frac{2}{7} = 2\frac{11}{12}$.
3. $\frac{8}{9} \div \frac{3}{11} = 1\frac{7}{27}$; $\frac{15}{16} \div \frac{1}{48} = 45$; $\frac{8}{25} \div \frac{1}{625} = 200$.
4. $\frac{13}{14} \div \frac{6}{13} = \frac{169}{84} = 2\frac{1}{84}$; $\frac{23}{31} \div \frac{19}{29} = 1\frac{78}{589}$.

214. The ratio of $\frac{p}{q}$ to $\frac{n}{d}$, which (188 and 213) is $\frac{pd}{nq}$, is (189) compounded of the ratios $\frac{p}{n}$ and $\frac{d}{q}$, and is, evidently, the product of these ratios.

Thus, we see that the quotient arising from the division of one fraction by another is *a compound ratio,* and is the product either of the direct ratio of the numerators and the inverse ratio of the denominators, or that of the dividend and the reciprocal of the divisor. Hence, the ratio of $\frac{pd}{nq}$ to $\frac{x}{y}$ is $\frac{pdy}{nqx}$, and is compounded of the two ratios $\frac{pd}{nq}$ and $\frac{y}{x}$; or of the three ratios $\frac{p}{n}$, $\frac{d}{q}$, and $\frac{y}{x}$; that is, a ratio compounded of any number of ratios *is the product of those ratios.*

Examples.

1. Find the ratio of $\frac{a}{b}$ to $\left(\frac{c}{d} \div \frac{n}{q}\right)$.

Answer, $\frac{adn}{bcq}$.

2. Find the ratio of $\left(\frac{p}{q} \div \frac{v}{x}\right)$ to $\left(\frac{x}{y} \div \frac{m}{n}\right)$.

Answer, $\frac{mpy}{nqv}$.

3. What is the ratio, compounded of the ratios A to B, B to C, and C to D? Answer, $\frac{A}{D}$.

Hence it is plain that, if we have any number of quantities, the ratio of the first to the last is compounded of the ratios, the first to the second, the second to the third, the third to the fourth, and so on to the last.

4. What is the ratio of a fraction to its reciprocal?

Answer, the duplicate ratio, or square of the fraction. Thus, the ratio of $\frac{p}{q}$ to $\frac{q}{p}$ is $\frac{p}{q} \div \frac{q}{p} = \frac{p}{q} \times \frac{p}{q} = \frac{p^2}{q^2}$.

5. What is the ratio of a fraction to the square of its reciprocal?

Answer, the triplicate ratio or cube of the fraction.

6. What is the ratio compounded of the ratios $\frac{a}{b}$ to $\frac{c}{d}$, $\frac{c}{d}$ to $\frac{d}{e}$, and $\frac{d}{e}$ to $\frac{e}{f}$? Answer, $\frac{af}{be}$.

7. What is the ratio compounded of the ratios to $\frac{2}{3}$ to $\frac{4}{5}$, $\frac{2}{7}$ to $\frac{5}{9}$, and $\frac{3}{4}$ to $\frac{3}{7}$? Answer, $\frac{3}{4}$.

215. As $\frac{p}{q} \div \frac{n}{d} = \frac{p}{q} \times \frac{d}{n} = \frac{pd}{qn}$; so $\frac{pd}{qn} \div \frac{x}{y} = \frac{pd}{qn} \times \frac{y}{x} = \frac{pdy}{qnx}$. Hence, $\frac{p}{q} \div \frac{n}{d} \div \frac{x}{y} = \frac{p}{q} \times \frac{dy}{nx} = \frac{pdy}{qnx}$. That is, to divide a quantity, successively, by any number of fractions, *we multiply that quantity by the product of their reciprocals.* In the same manner we find that $4 \div \frac{1}{2} \div \frac{1}{3} \div \frac{2}{5} \div \frac{3}{7} \div \frac{5}{9} = 252$.

216. A quantity divided by a fraction is always increased; *because the reciprocal, by which we multiply, is greater than a unit.* Also, it is increased *in the ratio of the numerator of the reciprocal to its denominator.* Wherefore, if this numerator be one unit greater than the denominator, *the quantity will be increased by such part of itself as is signified by the denominator.* Thus:

$$24 \div \tfrac{6}{7} = 24 \times \tfrac{7}{6} = (24 \times 1\tfrac{1}{6}) = 24 + 4 = 28$$
$$24 \div \tfrac{3}{4} = 24 \times \tfrac{4}{3} = (24 \times 1\tfrac{1}{3}) = 24 + 8 = 32$$
$$24 \div \tfrac{2}{3} = 24 \times \tfrac{3}{2} = (24 \times 1\tfrac{1}{2}) = 24 + 12 = 36$$

Divide 168, successively, by $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{6}{7}$, $\frac{7}{8}$, and find the sum of the quotients. Answer, 1200.

Division of Mixed Numbers.

217. Reduce mixed numbers to improper fractions; express whole numbers as fractions, and reduce compound fractions to simple ones: after which proceed as with ordinary fractions.

Examples.

1. $4 \div \frac{5}{6} = \frac{4}{1} \div \frac{5}{6} = \frac{4}{1} \times \frac{6}{5} = \frac{24}{5} = 4\frac{4}{5}$, and $\frac{5}{6} \div 4 = \frac{5}{6} \times \frac{1}{4} = \frac{5}{24}$.

Thus we see that the divisor, divided by the dividend, gives the reciprocal of the quotient, which serves as proof.

2. $87 \div 6\frac{3}{7} = \frac{87}{1} \div \frac{45}{7} = \frac{87}{1} \times \frac{7}{45} = \frac{203}{15} = 13\frac{8}{15}$.

Also, $6\frac{3}{7} \div 87 = \frac{45}{7} \times \frac{1}{87} = \frac{15}{203}$.

3. $7\frac{2}{7} \div 3\frac{3}{4} = \frac{68}{35} = 1\frac{33}{35}$, and $3\frac{3}{4} \div 7\frac{2}{7} = \frac{35}{68}$.

4. $\frac{3}{5}$ of $\frac{5}{8} \div \frac{2}{7}$ of $9\frac{1}{2} = \frac{3}{8} \div \frac{19}{7} = \frac{3}{8} \times \frac{7}{19} = \frac{21}{152}$.

5. $\frac{1}{3}$ of $9\frac{1}{15} \div \frac{6}{7}$ of $\frac{5}{6}$ of $2 = 2\frac{26}{225}$.

6. $758\frac{11}{16} \div 83\frac{3}{19} = 9\frac{3121}{25280}$.

7. $1435\frac{7}{8} \div 62 = 23\frac{79}{496}$.

8. $37296 \div 3\frac{11}{12} = 9522\frac{18}{47}$.

9. $1435\frac{7}{8} \div 23\frac{79}{496} = 62$.

10. $44983\frac{3}{11} \div 285\frac{7}{16} = 157\frac{29847}{50237}$.

218. When we divide a unit by a fraction, the quotient is *the reciprocal of that fraction.* Thus, $1 \div \frac{2}{3} = \frac{1}{1} \times \frac{3}{2} = \frac{3}{2}$. But the dividend is the product of the divisor and quotient; that is, the product of any fraction multiplied by its reciprocal is *a unit.*

Wherefore, *to divide or multiply by a fraction, is to multiply or divide by its reciprocal.*

To divide one whole number by the factors of another:

219. As the object of this method is, not only to expedite, but facilitate calculation, the number of factors is usually limited to two: because, where strict accuracy is required, and the dividend is prime to the factors, a greater number would render intricate the calculation of the final fraction.

Whenever, therefore, the divisor can be resolved into two factors, neither of which exceeds 12, *we first divide by one factor, and then divide the quotient by the other.*

For example, to divide 165369 by 45: As 45 is the product of 9 and 5, we first divide by one of these factors, no matter which, and then by the other, as follows:

$$\begin{array}{r}
5\underline{)\,165369\phantom{\tfrac{4}{5}}}\\
9\underline{)\,33073\tfrac{4}{5}}\\
3674\tfrac{7}{9}+\tfrac{4}{45}
\end{array}$$

Having divided by 5, the quotient is $33073\frac{4}{5}$.

Again, in dividing the integral part of this quotient, which is 33073, by 9, we have $3674\frac{7}{9}$; and, in dividing the fractional part $\frac{4}{5}$ by 9, (165,) we have $\frac{4}{45}$. We have, therefore, $3674\frac{7}{9}+\frac{4}{45}$ for the final quotient.

But, to have only one fraction, we add the two fractions $\frac{7}{9}$ and $\frac{4}{45}$ together. Now the denominator of $\frac{7}{9}$ is the last divisor, and the denominator of $\frac{4}{45}$, the product of the two divisors: consequently, as 9 is contained in 45 as often as expressed by the first divisor; to reduce $\frac{7}{9}$ to a fraction, the denominator of which shall be 45, we must multiply the numerator 7 by the first divisor 5, which gives $\frac{35}{45}$; then, adding the two fractions $\frac{35}{45}$ and $\frac{4}{45}$ together, we have $\frac{39}{45}$ for the sum. The numerator 39 of this sum is composed of the last remainder 7 multiplied by the first divisor 5, plus the first remainder 4: the denominator 45, being the product of the two divisors, or whole number divided by; and it is easy to see that the same kind of reasoning will hold, whatever may be the two remainders or divisors.

Hence, we have the following general rule: When there are two remainders, *multiply the last remainder by the first divisor, and to the product add the first remainder;* under this sum write the product of the two divisors, and reduce the fraction, if necessary, to its lowest terms.

The above example, performed by this rule, will stand thus:

$$\begin{array}{rl}
5\underline{)\,165369} & \\
9\underline{)\,33073}\ \ .\ \ .\ \ .\ \ 4 & \left.\right\} \dfrac{7\times5+4}{45}=\dfrac{39}{45}=\dfrac{13}{15}\\
3674\ \ .\ \ .\ \ .\ \ 7 & \\
 & \qquad\qquad 3674\tfrac{13}{15}\text{ quotient.}
\end{array}$$

It is evident that $3674\frac{13}{15}$ is the quotient of 165369 divided by 45; because (144) in dividing by 5, we take $\frac{1}{5}$; and, in dividing by 9, we take $\frac{1}{9}$ of $\frac{1}{5}$, which (211) is $\frac{1}{45}$.

When there is but one remainder, it is easy to perceive that if it belongs to the first division, it will take *the product of the two divisors* for a denominator; because it is a part of the dividend which has not yet been divided by either divisor, and that, if it belongs to the last division, it will take *the last divisor only,* for a denominator; because it has no more division to undergo.

Examples.

1. $237543 \div 25 = 9501\frac{18}{25}$.
2. $768539 \div 54 = 14232\frac{11}{54}$
3. $5970509 \div 63 = 94769\frac{62}{63}$.
4. $1084624 \div 56 = 19368\frac{2}{7}$.
5. $21165897 \div 49 = 431957\frac{4}{49}$.
6. $99392745 \div 35 = 2839792\frac{5}{7}$.

7. Divide 169281735, separately, by each of the numbers 24, 32, 36, 48, 81, and find the difference between the sum of the quotients and the dividend. Answer $146619403\frac{833}{864}$.

8. In the preceding question, what is the ratio of the sum of the quotients to the dividend; or, in other words, what part of the dividend is the sum of the quotients?
Answer $\frac{347}{2592}$.

9. What is the ratio of $146619403\frac{833}{864}$, to 169281735?
Answer $\frac{2245}{2592}$.

The ratios of each of two numbers to their sums are the complements of each other.

10. What is the ratio of 169281735 to $22662331\frac{31}{864}$?
Answer $\frac{2592}{347}$.

Here the dividend divided by the sum of the quotients gives its ratio to that sum. Hence, it is the product of the sum of the quotients and that ratio. Now, if the product of two numbers be divided by either, the quotient will be the other. Therefore, if we divide the dividend by its ratio to the sum of the quotients, we shall have that sum for the result. That is to say, *any quantity divided by its ratio to another quantity, will give that other quantity.*

But to divide by a fraction is the same as to multiply by its reciprocal. Wherefore, *any quantity multiplied by the reciprocal of its ratio to another quantity, will give that other quantity.*

BOOK III.

PRELIMINARY IDEAS—NUMERATION—ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF DECIMAL FRACTIONS, AND FRACTIONAL NUMBERS—REPEATING OR CIRCULATING DECIMALS.

SECTION X.

PRELIMINARY IDEAS—NUMERATION OF DECIMALS.

220. The word *decimal*, derived from the Latin word *decem*, ten, is applied to *that which is numbered by tens.* Hence, the scale of Natural Numbers (Sec. I.) is properly termed, *the decimal scale of Natural Numbers.* If, on the right of this scale, we place a comma, and consider a single unit, or unit of its right-hand order, to be an alt unit of the same scale continued downwards, that is, towards the right, it is evident (37) that this unit is greater than any definite number of figures which can stand on the right of the comma; consequently, those figures constitute *a fraction*, properly termed *a decimal fraction*, (often simply *a decimal.*)

221. As the whole scale is decimal, subject to a universal law, that ten units of any order constitute *one unit of the next order on the left*, or, inversely, that an alt unit of any order is equal to *ten units of the next order on the right*, it is plain, that every natural number, whole, fractional, or mixed, is *a decimal number*; notwithstanding which, *the figures on the right of the comma* are, by way of distinction, exclusively called *decimal figures*, or *decimals*; and those numbers alone, which contain, or, by reduction, are supposed to contain, orders on the right of the comma, are called *decimal numbers.*

222. Considering the scale of Natural Numbers to extend, in both directions, from the comma to infinity, it is now complete; the part on the left of the comma being *integral*, that is, (36,) *capable of expressing any whole number*, and that on the right *fractional*, that is, *capable of expressing any fraction*, or at

least (158) *its approximate value, within any assignable quantity.*

223. Beginning at the comma, if, on the right of it, we suppose a series of nines, thus, ,9999, &c., to extend ad inf., these (221) according to the universal law, are successively 9 *tenths*, 9 *hundredths*, 9 *thousandths*, 9 *ten-thousandths*, &c.; as vulgar fractions, $\frac{9}{10}$, $\frac{9}{100}$, $\frac{9}{1000}$, $\frac{9}{10000}$, &c.; that is, the denominator of each is *a unit followed by as many ciphers as there are decimal places*, counting *from the comma to the figure inclusive.* But any number of consecutive figures on the right of the comma, as well as in any other part of the scale, may (48) be read *as a number of units of the order of the right-hand figure of the number taken.* Thus, if we take, consecutively, one, two, three, four, &c. nines, we read, 9 *tenths*, 99 *hundredths*, 999 *thousandths*, 9999 *ten-thousandths*, &c. These, as vulgar fractions are $\frac{9}{10}$, $\frac{99}{100}$, $\frac{999}{1000}$, $\frac{9999}{10000}$, &c. Wherefore (48) the denominator of any decimal number, expressed as a vulgar fraction, is *a unit followed by as many ciphers as there are places of decimals in the number*, counting *from the comma to the right-hand place inclusive.*

224. The complements (205) of the fractions $\frac{9}{10}$, $\frac{99}{100}$, $\frac{999}{1000}$, &c. are $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c; that is, the complement of a decimal series of nines, beginning at the comma, is *a unit of the lowest order of the series;* also, (87,) the integral unit, on the left of the comma, is the alt of the series, and is first, second, third, fourth, &c.; that is, it is 10, 100, 1000, 10000, &c., according to the number of nines in the series. Hence, it is plain that a definite number of these nines can never equal a unit; but because the part lacking, or complement, is ten times less at each remove from the comma, it is easy to see that the ultimatum or limit of the series is *a unit.*

Let $S = ,9999$, &c. ad inf.

then $S = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}$, &c., ad inf.

Multiply both sides by 10. Then

$10S = 9 + (\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}$, &c , ad inf.) substitute S for its equivalent ($\frac{9}{10}$, $\frac{9}{100}$, $\frac{9}{1000}$, $\frac{9}{10000}$; &c. ad inf.)

Then we have $10S = 9 + S$.

Subtract S from both sides. Then $9S = 9$; and, dividing by 9, we have $S = 1$. Therefore, the series ,9999, &c. ad inf. $= 1$.

225. The alt, therefore, of a decimal is its denominator, when expressed as a vulgar fraction; and is *a unit followed by as many ciphers as there are places of figures in the decimal,* whether those places are supplied with significant figures or ciphers. Wherefore, to express a decimal as a vulgar fraction, we *first write the decimal as a whole number, for the numerator, and underneath we write its alt for denominator.* When there are ciphers between the comma and the highest order of the decimal, *we omit them in writing the numerator.* Thus,

,05; ,25; ,1250; ,0625, and ,00875, expressed as vulgar fractions, are $\frac{5}{100}$, $\frac{25}{100}$, $\frac{1250}{10000}$, $\frac{625}{10000}$, and $\frac{875}{100000}$.

These in their lowest terms are

$\frac{1}{20}$, $\frac{1}{4}$, $\frac{1}{8}$ $\frac{1}{16}$, and $\frac{7}{800}$

In the same manner the student will find that
,5 $=\frac{1}{2}$; ,75 $=\frac{3}{4}$. ,375 $=\frac{3}{8}$; ,1875 $=\frac{3}{16}$; ,03125 $=\frac{1}{32}$, and ,0015625 $=\frac{1}{640}$.

226. Ciphers, placed on the right of a decimal, do not alter its value, because, for every additional cipher in the decimal, or numerator, we have one more cipher in the alt, or denomitor, (165 and 225.) Thus:

,25 = ,250 = ,2500 = ,25000 = ,250000, &c., that is (165)
$\frac{25}{100}=\frac{250}{1000}=\frac{2500}{10000}=\frac{25000}{100000}=\frac{250000}{1000000}$, &c.

Hence, if the number of decimals in several decimal numbers be different, it may be rendered the same in each, without altering their value, *by placing ciphers on the right of those which have an inferior number, till they all have as many as that which has the greatest number.* Thus, instead of

25,3, ,416, ,7854, and 3,141592

we may write

25,300000; ,416000; ,785400, and 3,141592

227. To write, in its natural form, a decimal number, given in the form of a vulgar fraction, we first write the numerator as a whole number; then, beginning on its right-hand figure, we count towards the left as many figures, for decimals, (225,) as there are ciphers in the denominator, and, placing the comma on the left, we have the required decimal number. When the number of figures is not sufficient, we make it so, by placing ciphers on the left.

Thus, to express $\frac{83}{10000}$ in its natural form, we first write 83. Then, as there are four ciphers in the denominator, beginning on 3, we count towards the left, *one, two, three, four,* placing a cipher, as we say *three,* and another as we say *four;* and, prefixing the comma, we have ,0083 for the required decimal.

228. In the scale of Natural Numbers, (222,) the value of any figure is determined *by its distance from the comma.* Therefore, in any finite number, to remove the comma one, two, three, &c. places to the right, is the same as *to advance each figure one, two, three, &c. places towards the left;* that is, (36,) it is the same as *to multiply the number by* 10, 100, 1000, &c. Also, to remove the comma one, two, three, &c. places to the left, is *to advance each figure one, two, three, &c. places towards the right;* that is, (223,) *to divide the number by* 10, 100, 1000, &c.

Wherefore, to multiply a decimal number by 10, 100, 1000, &c., *we remove the comma one, two, three, &c. places towards the right.* Also, to divide the number by 10, 100, 1000, &c., *we remove the comma one, two, three,* &c. *places towards the left.* Thus:

$$,7854 \times 10 = 7{,}854;\quad ,7854 \times 1000 = 785{,}4.$$

If, in either direction, we do not find a sufficient number of places, *we supply the deficiency with ciphers.* Thus:

$$785{,}4 \times 100 = 78540;\ 1{,}6 \div 100 = ,016;\ ,7854 \div 1000 = ,0007854;\ \ ,7854 \times 1000000 = 785400$$

Numeration of Decimals.

229. The denominator of a decimal figure is (223) a unit followed by as many ciphers as there are places from the comma to the figure inclusive; and is also (225) the alt of a whole number containing the same number of places. We have, therefore, two methods by which to determine the order of any decimal figure; namely, we can point off the places of figures from the figure, considered as units, to the comma. Then, having read the figure, we pronounce the name of the alt or unit of the next higher grade with the usual denominating termination, *ths.* Or, beginning at the comma, we can pronounce on the places successively, *tenths, hundredths, thousandths, ten-thousandths, hundred-thousands, millionths, ten-millionths, hundred-millionths, billionths,* &c., till we reach the figure.

230. In the scale of Natural Numbers, a number of consecu-

tive figures, taken at pleasure, may (48) be read as a number of units of the order of the right-hand place of the part taken. To read a decimal fraction, therefore, we find, by either of the above methods, the name of the order of its right-hand place, which name we give to the whole, after reading it as a whole number. For example, to read the fraction ,5083020461079, having separated it into periods, as a whole number, we find that its highest order is units of trillions, and hence, its alt (87) *ten trillions;* consequently, the fraction is a number of *ten-trillionths.* We therefore read, five-trillions, eighty-three billions, twenty millions, four hundred and sixty-one thousand and seventy-nine *ten-trillionths.*

If all the figures of the above fraction, except the two last, were ciphers, we should still separate the places of figures into periods, find the denomination, as above, and read seventy-nine *ten-trillionths.* Express in words the following

Examples.

1. ,235
2. ,7089
3. ,6187
4. 7,08017
5. 91,002009

In reading a mixed decimal number, we insert the word *and* between the integral and decimal parts. The present example is read ninety-one, *and* two thousand and nine *millionths.*

6. 310,1065203
7. 70019,00700311
8. ,0034580275
9. 30375,000020067
10. ,81302000625031171811132569

231. From the above, we easily derive the method of writing the figures of a decimal fraction from its expression in words. Omitting the last word, *write the number enounced as a whole number.* Ascertain the number of ciphers in the alt, which is the last word divested of *ths.* Count from right to left, upon the figures of the number written, (and, if necessary, (227,) supply ciphers,) *as many places for decimals, as there are ciphers in the alt.* Then place a comma on the left, and the decimal is complete. Thus, to write ninety-five *millionths,* we first write 95 as a whole number. Then, as there are six ciphers in the alt, which is 1000000, we begin on 5 and count, towards the left, six places for decimals, supplying, of course, four ciphers; and have, 000095 for the required decimal.

We write two hundred and nineteen, and three hundred and twenty-seven *billionths*, thus:

219,000000327

first the integral part 219, with a comma on the right; then, as there are 9 ciphers in a *billion*, and only 3 figures in the number 327, we write 6 ciphers on the right of the comma, and on the right of these the number 327. Express in figures the following

Examples.

1. Nine units, and twenty-seven *thousandths.*

2. Six hundred and three units, and ninety-five *ten-thousandths.*

3. Eighteen units, and twelve thousand six hundred and four *millionths.*

4. Ten thousand and twenty-five *ten-millionths.*

5. Five millions, sixty thousand and eleven *billionths.*

6. Forty millions six hundred *units,* and ninety-seven thousand three hundred and fifty-eight *hundred-millionths.*

7. Sixty-four billions, forty-two millions, eleven thousand and seventeen *hundred-billionths.*

8. Write nine *units,* and twenty-seven *thousandths;* say how many *thousandths* this number contains, and how many times it would be lessened by changing it to *millionths.*

9. How many *ten-thousandths* are there in five hundred and four *units,* and sixty-seven *ten-thousandths?* and how many times would this number be lessened by changing it to *hundred-millionths?*

10. How many *millionths* are there in seventy thousand *units,* and thirty-six thousand and twenty-five *millionths?* and how many times would this number be lessened, by changing it to *hundred-billionths?*

SECTION XI.

ADDITION OF DECIMALS.

232. As the unit of any order, whether on the right or left of the comma, contains ten units of the next order on the right of it, we add numbers containing decimals, or decimals

alone, *in the same manner as whole numbers;* that is to say, we add units of the same kind together, and, for every ten in the sum of any order, we carry one to the next order on the left.

We also place the numbers under each other, *so that the commas may be under each other;* that is, so that units of the same order may be under each other, and place a comma in the sum *on the left of the tenths,* before we add the units.

For example, to add 54, 75,2, 95,56, and ,273, we place the numbers thus:

```
 54,
 75,2
 95,56
   ,273
-------
225,033
```

and having added the tenths, *we place a comma on the left under the other commas.*

We prove *by adding both ways,* as in whole numbers.

Examples.

1. 376,3 + 5,674 + ,23 + 150 = 532,204
2. ,36 + ,536 + 789,3 + 1,16 = 791,356
3. 373 + 25,25 + ,789 + 236,1 + 5,4 = 640,539
4. ,234 + ,5 + ,567 + ,462 + ,0005 = 1,7635
5. 532,204 + 791,356 + 640,539 + 1,7635 = 1965,8625

The scholar may prove this example by adding the numbers which compose the four preceding examples.

Subtraction of Decimals.

233. Place the numbers *so that units of the same order may be under each other:* subtract as in whole numbers, and place the comma in the remainder on the left of the tenths, or under the other commas, as in addition, (198.)

When one of the given numbers has more decimal figures than the other, we may (226) *render the number the same in both. Or, without placing ciphers on the right, we may subtract as though they were so placed.*

For example, to subtract 78,358 from 80,3, we place the numbers thus:

$$\begin{array}{r}80,300\\78,358\\\hline 1,942\end{array}\qquad \text{or thus :}\qquad \begin{array}{r}80,3\\78,358\\\hline 1,942\end{array}$$

Subtract as usual, and, placing the comma on the left of the tenths, we have 1,942 for the remainder. This and the succeeding examples may be proved both *by addition and subtraction.*

Examples.

1. 652,34 — 58,603 = 593,737
2. 1009,3 — 9,6389 = 999,6611
3. 711,823 — 70,9 = 640,923
4. 21000 — ,5001 = 20999,4999
5. 1 — ,99951807324 = ,00048192676

6. What is the difference between the sum of the answers to the five preceding examples, and 100000?

Answer: Seventy-six thousand, seven hundred and sixty-six units, and seventeen billions, eight hundred and fifty-one millions, eight hundred and seven thousand, three hundred and twenty-four *hundred-billionths.*

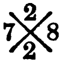
Multiplication of Decimals.

234. The denominator of a decimal number (225) is a unit, followed by as many ciphers as there are decimal places in the number. Therefore, (206,) the denominator of the product of two such numbers is a unit, followed by as many ciphers as there are decimal places in both numbers: that is, (227,) *there are as many decimal places in the product as in both factors.* Hence, the following rule: Place the numbers and multiply them as whole numbers; and, in the product, point off for decimals as many places as there are decimal places in both factors.

When the number of figures in the product is not sufficient, we *complete it by placing ciphers on the left, and then prefix the comma.*

To multiply 7,54 by 7,1, without regard to the comma, we place the numbers thus:

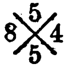
$$\begin{array}{r}7,54\\7,1\\\hline 53,534\end{array}$$



and having multipled and proved as in whole numbers, as there are three decimal places in the two factors, we point off the three right-hand places of the product.

Again, we multiply, as in whole numbers, ,1853 by ,013, thus :

$$\begin{array}{r} ,1853 \\ ,013 \\ \hline ,0024089 \end{array}$$



and, as there are 7 decimals in the two factors, and but 5 figures in the product 24089, we place two ciphers on the left, and prefix the comma.

The student may prove the following multiplications, by multiplying both ways, and by casting out the nines :

Examples.

1. ,153 × ,82 × ,017 = ,00213282.

$$\begin{array}{r} ,153 \\ ,82 \\ \hline 306 \\ 1224 \\ \hline ,12546 \\ ,017 \\ \hline ,00213282 \end{array} \qquad \qquad \begin{array}{r} ,82 \\ ,017 \\ \hline ,01394 \\ ,153 \\ \hline 4182 \\ 20910 \\ \hline ,00213282 \end{array}$$

2. 7,813 × 3,69 × 2,17 =
3. 11,917 × 1261,4 × 2,84 =
4. ,05418 × 4,816 × 75,15 =
5. 1926,48 × 17,34 × 1,632 =
6. ,00625 × ,039 × ,05 × ,8 =

7. Required the difference between the sum of the products of the six preceding examples, and one million.

Answer. Nine hundred and two thousand, seven hundred and nine units; and four hundred and twenty-six millions, seven hundred and thirty-two thousand, four hundred and ninety-eight *billionths.*

235. When the product is only required to a proposed degree of exactness, the work may be abridged by the following method, given by Bezout : We reverse the order of the figures of the multiplier and write it under the multiplicand, so that

its unit figure may stand *under the second place on the right of that to which the product is required.* We then multiply by each figure of the multiplier, beginning with *the figure under which it stands,* and place the first figure of each new product *under that of the preceding one.* Having added the products, we *suppress the two right-hand figures,* observing however, to *increase by a unit the last of those which remain* when the suppressed figures exceed 50. Lastly, we *point off the places of the proposed limit.*

Required, the product of ,2345678 × 85,276 within a thousandth.

$$\begin{array}{r}
,2345678\\
67258\\
\hline
1876536\\
117280\\
4690\\
1638\\
138\\
\hline
20,002\cancel{8}\cancel{2}
\end{array}$$

As the figures suppressed exceed 50, we increase the next figure 2 by a unit, and have 20,003 for the product within *a thousandth.* The true product is 20,0030037128.

In placing the unit figure of the multiplier under the second grade below the proposed limit, the product is of the order of that grade. Also, by the inverted order of the figures, the next lower order of the multiplicand is multiplied by a tenfold higher order of the multiplier; and the next higher order of the multiplicand by a tenfold lower order of the multiplier, and so on in succession. Therefore, *all the products are of the same order,* which accounts for placing their first figures under each other.

Again, because the part rejected cannot equal a unit of the order multiplied, the product of this part by any figure cannot equal a unit of the next higher order: therefore, each product is within *a unit of the next order on the right of the proposed limit.* Hence, the rule, as an approximation, may in all ordinary cases be relied on.

When there is no unit figure in the multiplier, *place 0 in its stead.* Also, when there are not enough of decimal places in the multiplicand, *supply the deficiency with ciphers.*

Find the product of ,227538917 × ,5664178 to the seventh decimal inclusive.

$$\begin{array}{r}
,227538917 \\
87146650 \\
\hline
113769455 \\
13652334 \\
1365228 \\
91012 \\
2275 \\
1589 \\
176 \\
\hline
1288820\cancel{0}\cancel{0}
\end{array}$$

Product ,1288821.

The student may prove the following examples by multiplying in the ordinary way:

Examples.

1. Required, the product of 376,273495 multiplied by 2,73486, correct to *one-thousandth.* Answer, 1029,055 +.
2. Required, the product of 83,4679215 × 67,89341, correct to *a hundred-thousandth.* Answer, 5666,92182.
3. Required, the product of 75,82344 × ,196497, true to five decimals. Answer, 14,89908.
4. Required, the product of ,7358462199 × ,324162549, correct to nine decimals. Answer, ,238533786 +.

Division of Decimals.

236. When the divisor is a whole number, the quotient, (150) being of the same order as the dividend, *must have the same number of decimals.*

237. When the divisor is not a whole number, we *place as many ciphers on the right of the dividend, when integral, or remove its comma, when decimal, (annexing ciphers if necessary,) as many places to the right, as there are decimal places in the divisor, in which we then suppress the comma.* Both numbers being thus (228) multiplied by the same number, the quotient (165) is not altered. The divisor being now a whole number, we divide as usual, and point off for decimals, in the quotient *as many places as there are decimal places in the dividend.*

When the divisor is rendered integral, ciphers on the left of its highest grade are effaced. But ciphers between the comma and the highest order of the dividend, though neglected in the operation, are retained to ascertain the place of the comma in the quotient. Also, we place *any number of ciphers at pleasure,* as decimal places, on the right of the dividend, when it does not contain the divisor.

Examples.

1. 5236 ÷ ,128 = 5236000 ÷ 128 = 40906,25

We here first multiply both numbers by 1000, which renders the divisor integral, and (165) does not affect the quotient. Then, to avoid confusion in distinguishing the annexed ciphers in integral, from those in decimal places, we at once determine the place of the comma, thus: As the part 523, which is tens of thousands, contains the divisor, the quotient figure 4 is (150) tens of thousands; that is, there will be 5 integral figures in the quotient: therefore, having found these, we place the comma on the right.

2. 6,375 ÷ 85 = ,075

Having divided, as in whole numbers, and found 75 for the quotient, as there must be (236) as many decimal figures in the quotient as in the dividend, we place a cipher on the left of 75, and, prefixing the comma, we have ,075 for the true quotient.

238. When the divisor and dividend have each the same number of decimal places, *the comma in both may be suppressed:* because the equimultiples of any two numbers (165) give the same quotient as the numbers do. Thus:

6,375 ÷ ,075 = 6375 ÷ 75 = 85, (237.)

239. If, after we have rendered the divisor integral, the dividend does not contain it, *we place a comma in the quotient, and also on the right of the dividend, if integral. We then place just as many ciphers on the right of the dividend as will make it contain the divisor; and, on the right of the comma, in the quotient, as many ciphers, less one, as there are now decimal places in the dividend;* because the quotient figure itself will occupy the place of the last cipher in the dividend, which (150) is its true order.

Examples.

1. ,128 ÷ 81,92 = 12,8 ÷ 8192 = 12,800 ÷ 8192

Having thus prepared the numbers, as the dividend now contains three decimals, we write ,001 in the quotient, which, when complete, is ,0015625.

2. ,128 ÷ ,0015625 = ,1280000 ÷ ,0015625 = 1280000 ÷ 15625 = 81,92

Having rendered the numbers integral, as the first partial dividend is tens, we know that there will be two integral figures: as soon, therefore, as we obtain these, we place a comma on the right and continue the operation.

240. Ciphers on the right of an integral divisor may always be suppressed *by removing the comma in the dividend one place to the left for every such cipher.*

Examples.

1. 2248318,477 ÷ 970000 =
224,8318477 ÷ 97 = 2,3178541
2. 27445863,6 ÷ 36000 =
27445,8636 ÷ 36 = 27445,8636 ÷ (6 × 6) =
4574,3106 ÷ 6 = 762,3851

241. When both numbers terminate with the decimal form of some well-known fraction, having, in its denominator, no factor prime to 10, such as ,25 and ,75, which are $\frac{1}{4}$ and $\frac{3}{4}$; ,125; ,375; ,625; and ,875; which are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$; or, ,0625; ,1875; ,3125; ,4375; ,5625; ,6875; ,8125, and ,9375; which are $\frac{1}{16}$, $\frac{3}{16}$, $\frac{5}{16}$, $\frac{7}{16}$, $\frac{9}{16}$, $\frac{11}{16}$, $\frac{13}{16}$, and $\frac{15}{16}$; the work may often be greatly facilitated *by multiplying both numbers by the denominator of the known fraction*, in doing which we recognize, without calculation, *the numerator of the fraction, for the product of the terminating figures.*

Examples.

1. 536,25 ÷ ,75 = 2145 ÷ 3 = 715

We here consider ,25 and ,75 as the numerators of the fractions which they represent. Wherefore, in multiplying the dividend by 4, we begin on the left of ,25, and say, 4 times 6 is 24, and 1 is 25, &c.

2. $45616{,}875 \div {,}625 = 4561{,}6875 \div {,}0625 = \frac{72987}{1} =$ 72987

By simply changing the place of the comma in both numbers, or conceiving it to be changed, one place to the left, which (165) does not affect the quotient, we here recognize ,6875 and ,0625 as the numerators, 11 and 1, of their equivalents. Then, beginning on the left of ,6875, we multiply by 16, saying, 6 times 1 is 6, and 11 is 17, *seven* and *go* 1; then 6 times 6 is 36, and 1 is 37, and 1, on the right of 6, is 38; *eight* and *go* 3, and so on.

3. $136581{,}25 \div {,}9375 = 2185300 \div 15 =$
$437060 \div 3 = 145686{,}66$, &c.

Without regard to the comma, we begin on the left of 8125, and say, 6 times 5 is 30, and 13 is 43; three and go 4: then, 6 times 6 is 36, and 4 is 40; and 5 is 45, and so on; taking care to multiply the result by 100, because the comma is 2 places to the right.

4. $23587{,}40625 \div {,}4375 = 53914{,}0714285 +$

To divide by a fraction is *to multiply by its reciprocal.* Therefore, to divide by $\frac{7}{16}$ is to multiply by $\frac{16}{7} = 2\frac{2}{7}$.

Proof.

$$\begin{array}{rl} 23587{,}40625 & \left.\begin{array}{l}\\ \\ \end{array}\right\} \text{twice the number.} \\ 23587{,}40625 & \\ 6739{,}25892 & \frac{2}{7} \text{ of the number.} \\ \hline 53914{,}07142 + . & \end{array}$$

5. $358127{,}95625 \div 68{,}75$. Multiply both by $\frac{16}{100}$. Then $57300{,}473 \div 11 = 5209{,}13390909$, ad inf.

The student has seen (176) that, in dividing by 11, the *primary alternate figures, first, third, fifth,* &c. are remainders; and the *secondary*—that is, the *second, fourth, sixth,* &c.—are undecimal complements of the remainders; also, that when the sums of primary and secondary are equal, the number is a multiple of 11. Hence, if the sum of the primary is the greater, *the difference is the true remainder;* but, if that of the secondary is the greater, *the undecimal complement of the difference is the remainder.* Now, in the number 57300,473, as the sum of the primary is 14, and of the secondary 15, the complement of the difference 1, that is to say, 10 is the true remainder. Beginning with the right-hand figure 3, we subtract the remainder, saying, 10 from 13, *three*. This we write as the right-hand figure of the quotient. Then, before we again subtract, we add 1 to this 3, for the 10 previously added, and say, 4 from 7, *three;* which is the next figure of the quotient towards the left. Then, 3 from 4, *one*, which is the *third*, on the left of which we place the comma. Then, 1 from 10, *nine;* 10 from 10, *nought;* 1 from 3, *two;* 2 from 7, *five;* and 5 from 5, *nought;* which last we do not write. We thus obtain the quotient

$$5209,133\tfrac{10}{11} = 5209,1339090 \text{ ad inf.}$$

This, which is merely the countermarch of the operation of multiplying by 11, (126,) we certainly do not give for its practical utility, but to accustom the student promptly to revert to previously acquired principles.

6. $77892,541875 \div 812,5 =$
$1246,28067 \div 13 = 95,86774 + \tfrac{5}{13}$

The scholar should, without again writing the numbers, divide thus: 13 in 124, *nine* times, and 7 over; writing 9 in the quotient: 13 in 76, *five* times, and 11 over; writing 5 and placing a comma on the right: then 13 in 112, *eight* times, and 8 over: 13 in 88, *six* times, and 10 over: 13 in 100, *seven* times, and 9 over: 13 in 96, *seven* times, and 5 over: 13 in 57, *four* times, and 5 over.

242. With an integral divisor, the quotient must (236) have *as many decimals as the dividend.* Now, if we divide the divisor, (165,) we multiply the quotient; therefore, (237,) for every decimal in the divisor, we must move the comma in the quotient one place towards the right. Hence, there will ultimately remain in the quotient *a number of decimals equal to the difference between the number in the dividend and the number in the divisor.*

Again, the product of two decimal numbers has (234) as many decimals as both factors: therefore (60) the dividend has *as many as the divisor and quotient.* Hence, we have the following general rule: Separate by a comma a number of the right-hand figures of the quotient, equal to *the number by which the decimal figures of the dividend exceed those of the divisor,* and where the number of figures is not sufficient, *supply the deficiency with ciphers,* and prefix the comma. The previous methods are perhaps preferable.

Examples.

1. ,0658 ÷ 3256 = ,000020208 +
2. ,654 ÷ ,002191 = 298,4938 +

As the terminating figures 38 are much nearer to 40 than 30, if we required but 3 decimals, we should write 298,494 —. The sign — indicates that the quotient is somewhat too great.

3. 23 ÷ ,71875 = 32; and $\frac{61}{16}$ = 3,8125
4. 149,172 ÷ ,496 = 300,75
5. 63261 ÷ 64,8 = 976,25
6. 52095,10875 ÷ 6358,49 = 8,193 +
7. 3,141593 ÷ ,7854 = 3,9999 +
8. 149 ÷ 15,36 = 9,7005208 +
9. 21,4 ÷ ,536 = 39,92537 +
10. ,2673 ÷ 39 = ,00685384615 +
11. ,7103 ÷ 8549 = ,0008386 —
12. 1,2193 ÷ 136324 = ,000089440597 +

243. The work is often contracted *by performing the subtraction, as we multiply each figure of the divisor*, and writing, instead of the product, *the remainder only.*

Thus, to divide 473804 by 899:

$$\begin{array}{l} 899)\,473804\,(527\tfrac{31}{899} \\ \quad\quad\;\, 2430 \\ \quad\quad\;\;\; 6324 \\ \quad\quad\quad\;\; 31 \end{array}$$

Having found the quotient figure 5, by saying 9 in 47, we say, 5 times 9 is 45: then, 45 from 8 being impossible, we say, 45 from 48, *three*, and write 3 under 8. Then, having added 4 tens to 8, we add 4 units to the next product: thus, 5 times 9 is 45, and 4 is 49. Then, 49 from 53, *four*, which we write under 3. Having added 5 tens, we add 5 units to the next product, thus: 5 times 8 is 40, and 5 is 45. Then, 45 from 47, *two*, which we write under 7. To the right of the remainder 243, we bring down the figure 0, and have 2430 for the next partial dividend.

With this we proceed in like manner, always adding *as many tens as will enable us to subtract, and* (85) 1 *to the next product for each ten added to the upper figure.*

244. When we only require the quotient within a unit, to contract the calculation, we suppress of the right-hand figures of the dividend *as many, less one, as there are figures in the*

divisor. Having exhausted this dividend, if there is no remainder, we place, on the right of the quotient, *a cipher for each figure suppressed in the dividend.* If there is a remainder, we consider it the new partial dividend. We then *strike out, by a bar,* as in cancelling, *the right-hand figure of the divisor,* always taking *the remainder* for a new dividend, and suppressing, at each division, *the right-hand figure of the divisor,* till the divisor is exhausted.

To find the quotient of 9968592476 ÷ 52493, within a unit, we suppress the figures 2476 of the dividend, and divide 996859 by 52493, contracting the work as above, thus:

$5\cancel{2}\cancel{4}\cancel{9}\cancel{3}$) 996859 (189903
471929
51985
4744
19
4

We first divide by the whole divisor till we have exhausted the dividend, and have 18 for the quotient and 51985 for the remainder. We then bar the figure 3 in the divisor, and dividing by 5249, we have 9 for the quotient and 4744 for the remainder. Again, because 9 is nearly a unit of the next order to the left, in barring it, we add a unit to the next figure; and, dividing by 525, we have 9 for the quotient and 19 for the remainder. Barring again, we divide by 52, and have 0 for the quotient and 19 for remainder. Barring for the last time, we divide 19 by 5, and have 3 for the quotient and 4 for the remainder. Thus, then, we have obtained 189903 for the quotient, within a unit. The true quotient is 189903 $\frac{14297}{52493}$.

245. If, after barring the right-hand figure of the divisor, the partial dividend is too small to contain it, *we place a cipher in the quotient,* and again bar the right-hand figure of the divisor. If the dividend is still too small to contain the divisor, *we place another cipher in the quotient,* bar another figure of the divisor, and so on.

Also, when the division becomes exact, before the divisor is exhausted, we place *as many ciphers, less one,* in the quotient *as there are figures remaining in the divisor.*

When the figure which we bar in the divisor, also when the left-hand figure of the part suppressed in the dividend, exceeds 5, *we increase by a unit the next figure on the left.*

To divide 1911740575899 by 546118, so that the quotient may be within a unit, we suppress the figures 75899 of the dividend; and, adding 1 to 5, the next figure on the left, we have 19117406, which we divide by 546118, thus:

$$\begin{array}{l} 5\overset{\circ}{4}\overset{\circ}{6}\cancel{1}\cancel{1}\cancel{8})\,19117406\,(3500600 \\ \qquad\quad\;\; 2733866 \\ \qquad\qquad\quad 3276 \\ \qquad\qquad\quad {*}{*}{*}{*} \end{array}$$

Having found the quotient 35 and the remainder 3276, as this does not contain the divisor 54612, after barring the figure 8, we place 0 in the quotient, and bar the next figure of the divisor; but, as 3276 does not contain 5461, we place another cipher, and bar the next figure. Then, as the divisor is contained exactly 6 times, we place 6 in the quotient, and, on the right, two ciphers for the two remaining figures, which we should have barred.

The true quotient being $3500599\frac{451217}{546118}$, the quotient, obtained by contraction, is not only within a unit, but within *one-fifth* of a unit.

246. If, after the suppression of the right-hand figures, the first dividend does not contain the divisor, we *suppress, at once, as many as are necessary of the right-hand figures of the divisor, in order that it may be contained.*

To have the quotient of 2349658 ÷ 49786, within a unit, we first suppress the figures 9658 of the dividend, increasing, by a unit, those which remain. Then, as 235 does not contain 49786, we at once suppress the figures 786, and increase, by a unit, those which remain. We then divide 235 by 50, thus:

$$\begin{array}{l} 5\cancel{0})\,235\,(47 \\ \qquad\;\; 35 \\ \qquad\;\; {*}{*} \end{array}$$

and have 47 for the quotient, which is within *one-fifth* of a unit; the true quotient being $47\frac{9716}{49786}$

247. With an integral divisor, the quotient is *a number of units of the order of the right-hand figure of the dividend:* consequently, we can, by the same method, find *the quotient within a unit of any order of decimals;* namely, by assuming as many decimals on the right of the dividend as will reduce it to the required order.

Suppose we would have, within a *ten-thousandth*, the quotient of 7258 ÷ 6319. As the dividend would require 4 decimals, and the divisor that we should suppress 3 figures, we place one cipher on the right of 7258; and, proceding as before, we point off, in the quotient, 4 figures for

$$\begin{array}{rrl} 631\cancel{9}) & 72580 & (1{,}1486 \\ & 9390 & \\ 6\cancel{3}\cancel{1} & 3071 & \\ & 543 & \\ & 39 & \\ & 3 & \end{array}$$

decimals. The true quotient being 1,148599 +, the quotient obtained by contraction is within not only a *ten-thousandth*, but a *millionth* of a unit.

248. When there are decimals in one or both of the given numbers, we may render them integral, (226 and 238,) and operate as in the preceding example. Thus, to find the quotient of 562,49 ÷ ,3927, within a thousandth, we have (238) 5624900 ÷ 3927. Then, as the dividend would require 3 decimals, and the divisor the suppression of 3 figures, the numbers remain as they are.

$$\begin{array}{rrl} 392\cancel{7}) & 5624900 & (1432{,}366 - \\ & 16979 & \\ & 12710 & \\ & 9290 & \\ 3\cancel{9}\cancel{2} & 1436 & \\ & 257 & \\ & 23 & \end{array}$$

As the figure 3, increased by a unit, and consequently too large, is contained in the remainder 23 nearly 6 times, we place 6 in the quotient with the sign —.

The above is easily applied to the reduction of a vulgar fraction to a decimal having a limited number of figures. Thus, if we would have the value of $\frac{375}{629}$ within a ten-thousandth, as the dividend requires 4 ciphers, and the divisor the suppression of two figures, we divide 37500 by 629, as usual, and have ,5962 — for the quotient.

$$\begin{array}{rrl} 62\cancel{9}) & 37500 & ({,}5962 - \\ & 6050 & \\ 6\cancel{2} & 889 & \\ & 11 & \end{array}$$

The true quotient is ,59618 +.

249. By the suppression of its units figure, the divisor, each time, becomes *of a tenfold higher order*, and requires the quotient to be *of a tenfold lower order* to give an invariable product. Each new product and new dividend is therefore *of the same order;* that is, all the products and dividends are of the order *of the right-hand figure of the first-prepared dividend.*

The division of the suppressed part of the dividend by the divisor, could not give a unit in the quotient; that is to say, the quotient cannot, on account of this suppression, be a unit too small.

Again, the suppression of the figures of the divisor tends to increase the quotient; that is, to counteract the error occasioned by the suppression of part of the dividend. Also, as we take the divisor within half a unit, increasing, when necessary, its right-hand figure; the error, which cannot exceed about half a unit of the next order on the left of the denominating, or right-hand order, is *sometimes plus and sometimes minus.*

Hence, we infer, that for all ordinary purposes, where only a small number of decimals is required, the method, as an expedite approximation, may be relied on.

Required the quotient of ,527694735 ÷ ,23748, within a *ten-thousandth.*

Removing the comma 5 places to the right, we have 52769,4735 ÷ 23748, (237 :) then, as the dividend is a number of ten-thousandths, and the divisor requires the suppression of 4 figures, we divide thus:

$$\begin{array}{rrl} 2374\cancel{8}) & 52769 & (2,2220 \\ 23\cancel{7}\cancel{5} & 5273 & \\ & 523 & \\ 2\cancel{4} & 49 & \\ & 1 & \end{array}$$

The divisor 24 being too great and the last fraction $\frac{1}{2}$, we may, if we please, write the quotient thus:

2,2211 —. The true quotient is 2,222059 +.

250. When the denominator of a fraction, which we would reduce to a decimal, is the product of two or more convenient factors, we divide as in article 219; except that here we can carry the division, where it is not exact, to any proposed degree of approximation, without noticing the final remainders. For example, to find the decimal value of $\frac{120}{121}$, within a *ten-thousandth*, we proceed thus:

$$\begin{array}{r} 11)\underline{120,0000} \\ 11)\underline{10,9090} \\ ,9917+ \end{array}$$

The quotient ,9917 is (37) within a *ten-thousandth.*

Again, to have the value of $\frac{119}{495}$, within a *ten-thousandth*, as it is easy to perceive, from what has been said on the properties of numbers, that $495 = 5 \times 9 \times 11$, we proceed thus:

$$\begin{array}{r} 5)\underline{119,0000} \\ 9)\underline{23,8000} \\ 11)\underline{2,6444} \\ ,2404 \end{array}$$

As the next figure would be a cipher, the quotient ,2404 is not only within a *ten-thousandth,* but within a *hundred-thousandth.*

Examples.

1. $\frac{1}{2}=,5$; $\frac{1}{4}=,25$; $\frac{3}{4}=,75$; $\frac{1}{8}=,125$.
2. $\frac{3}{8}=,375$; $\frac{5}{8}=,625$; $\frac{7}{8}=,875$; $\frac{1}{16}=,0625$.
3. $\frac{3}{16}=,1875$; $\frac{5}{16}=,3125$; $\frac{7}{16}=,4375$.
4. $\frac{9}{16}=,5625$; $\frac{11}{16}=,6875$; $\frac{13}{16}=,8125$; $\frac{15}{16}=,9375$.

The above fractions being of frequent recurrence in business, the scholar should remember their decimal equivalents.

5. $\frac{604}{605}=,998347107438016+$.
6. $\frac{791}{792}=,998737373+$.

Let the following divisions be performed according to the method of contraction, art. 248 and following:

7. Required the value of $\frac{31}{899}$, within a *ten-thousandth.*
Answer, ,0345—.
8. Required the value of $\frac{161}{899}$, within a *hundred-thousandth.*
Answer, ,17909—.
9. Required the value of $\frac{18}{179}$, within a *hundred-thousandth.*
Answer, ,10055.
10. Required the value of $\frac{868}{899}$, within a *ten-thousandth.*
Answer, ,9655.
11. Required the value of $\frac{67019}{67132}$, within a *hundred-thousandth.*
Answer, ,99832.
12. Required the value of $\frac{113}{67132}$, within a *hundred-thousandth.*
Answer, ,00168.

SECTION XII.

REPEATING DECIMALS.

251. When the value of a fraction cannot be exactly expressed in decimals, we find, by sufficiently extending the division, *a periodical return of the same figures in the quotient* without end, whence they are often called *periodical, circulating, or infinite* decimals. Thus, in finding the value of $\frac{7}{11}$, we have, 6363, &c.; where, in continuing the division, the figures 6 and 3 will succeed each other without end. The reason of this will appear evident as follows: the remainder must be one of the numbers 1, 2, 3, &c., but cannot equal the divisor: therefore, if we have a different remainder at each division, *we cannot perform as many divisions as there are units in the divisor, without falling upon some remainder that we have had before,* in which case the quotient figures will return in the same order.

252. The figures which repeat, constitute what is called *the period.* Thus, in the expression ,6363, &c., the period is 63. In the expression ,568568, &c., the period is 568.

To distinguish the period with facility, if it contains only one figure, a point is placed over this figure: thus, ,1111, &c. is written $,\dot{1}$: and 333, &c. is written $,\dot{3}$. If it contains more figures than one, it is distinguished by two points, one of which is placed over the first figure, and the other over the last, thus: ,6363, &c. is written $,\dot{6}\dot{3}$; and ,568568, &c. is written $,\dot{5}6\dot{8}$.

253. As a unit, in any place towards the left, is composed of a number expressed by as many nines as there are places on the right of it, plus 1; it is plain that in dividing a unit by 9,99,999, &c., as the remainder will always be a unit, the only significant figure in the continued quotient will also be a unit. Thus we see that $\frac{1}{9} =$,11111, &c.; $\frac{1}{99} =$,0101, &c.; $\frac{1}{999} =$,001001, &c.

Now we may consider the periodical decimal fraction ,6363, &c.; or $,\dot{6}\dot{3}$ as the product of ,0101, &c., or $,\dot{0}\dot{1}$ multiplied by 63. But, $,\dot{0}\dot{1} = \frac{1}{99}$; therefore, $,\dot{6}\dot{3} = \frac{1}{99} \times 63 = \frac{63}{99}$. In the same manner we find that *any periodical decimal fraction is equivalent to the vulgar fraction of which the numerator is the period, and the denominator as many nines as there are* *places of figures in the period.* Thus, $\dot{5}6\dot{8} = \frac{568}{999}$; $,\dot{0}1\dot{6} = \frac{16}{999}$, and $,\dot{0}03\dot{1} = \frac{31}{9999}$.

254. It frequently happens that the period does not commence with the first decimal figure, in which case, we consider the figures on the left of the period as units. On the right of these units, we place the vulgar fraction equivalent to the period, and reduce the mixed number, thus formed, to an improper fraction. Lastly, on the right of the denominator we place *as many ciphers as there are decimal places between the period and the comma.*

For example, if we would reduce ,325656, &c., or $,32\dot{5}\dot{6}$ to its equivalent vulgar fraction, as the period is equal to $\frac{56}{99}$, the fraction $,32\dot{5}\dot{6}$ is $,32\frac{56}{99}$.

Suppressing the comma, and reducing $32\frac{56}{99}$ to an improper fraction, we have $\frac{3224}{99}$. But, in suppressing the comma, we render the number 100 times its former value; we must therefore divide $\frac{3224}{99}$ by 100, which is done (55 and 165) in placing two ciphers on the right of the denominator 99. We have therefore $,32\dot{5}\dot{6} = \frac{3224}{9900}$.

It is easy to see that when the figures between the comma and the period are ciphers, we may proceed at once to express the period by its equivalent fraction, and place those ciphers on the right of its denominator: also that, in many cases, *we may reduce the fractional equivalents to lower terms.* For example, $,00\dot{2}47\dot{5} = \frac{2475}{999900} = \frac{225}{90900} \times \frac{25}{10100} = \frac{1}{404}$.

Let this example be proved by reducing $\frac{1}{404}$ to a decimal.

Examples.

1. $,\dot{6}2\dot{1} = \frac{23}{37}$, and $,2\dot{2}\dot{7} = \frac{5}{22}$
2. $,\dot{0}3\dot{7} = \frac{1}{27}$, and $,0\dot{0}3\dot{7} = \frac{1}{270}$
3. $,\dot{0}2\dot{7} = \frac{1}{37}$, and $,00\dot{0}2\dot{7} = \frac{1}{3700}$
4. $,\dot{7}2\dot{9} = \frac{27}{37}$, and $,007\dot{6}\dot{4} = \frac{757}{99000}$
5. $,\dot{0}09\dot{9} = \frac{1}{101}$, and $,000\dot{2}47\dot{5} = \frac{1}{4040}$

255. When two figures repeat, the denominator of the fractional equivalent (253) is 99. But $99 = 9 \times 11$; wherefore, if the two figures which repeat be a multiple of 9, as the fac-

tor 9 in the numerator will cancel the same factor in the denominator, the fractional equivalent, in its lowest terms, will have 11 for its denominator; that is to say, will be *a number of elevenths.* Thus, $,\dot{2}\dot{7} = \frac{3}{11}$. Now the figure 2, in the place of tens, is 2 (9 + 1;) that is, it is 2 nines plus 2; and this remainder 2, with the 7 units, makes one nine more. Therefore, when the sum of two figures, which repeat, is 9, *add a unit to the left-hand figure, and the sum will be the number of elevenths equivalent to the repeating decimal fraction.*

Hence also, to have the decimal value of a proper fraction, having 11 for its denominator, we have only to *multiply its numerator by* 9, and the product will be the decimal period. Thus, to find the value of $\frac{5}{11}$, we say $5 \times 9 = 45$; therefore, $\frac{5}{11} = ,\dot{4}\dot{5}$.

$\frac{1}{11} = ,\dot{0}\dot{9}$; $\frac{2}{11} = ,\dot{1}\dot{8}$; $\frac{3}{11} = ,\dot{2}\dot{7}$; $\frac{4}{11} = ,\dot{3}\dot{6}$; $\frac{5}{11} = ,\dot{4}\dot{5}$
$\frac{6}{11} = ,\dot{5}\dot{4}$; $\frac{7}{11} = ,\dot{6}\dot{3}$; $\frac{8}{11} = ,\dot{7}\dot{2}$; $\frac{9}{11} = ,\dot{8}\dot{1}$; $\frac{10}{11} = ,\dot{9}\dot{0}$.

256. If a fraction be given in its lowest terms, its decimal value will be finite; which means exact or definite, *when its denominator is a measure of* 10 or of some power of 10; that is, of one of the numbers in the scale 10, 100, 1000, &c. When the denominator is prime to 10, and consequently, to every number in the scale 10, 100, 1000, &c., *the decimal value will be infinite;* that is, (251,) it will be *a repeating decimal.* For, in placing a cipher continually on the right of each remainder, we do the same as to multiply the dividend by 10, 100, 1000, &c.; consequently, if the denominator is a measure of any number in this scale, it must be a measure of the numerator when multiplied by that number, and hence the quotient will be exact. Also, when the denominator is prime to the numbers 10, 100, 1000, &c., as it is prime to its numerator, it will (180) be prime to the product of the numerator multiplied by any of these numbers, and the quotient (251) will, consequently, be a repeating decimal. Now the denominator cannot measure any of the numbers 10, 100, 1000, &c. if it contains any prime factor which is not common to those numbers; that is, if it contains any prime factor other than 2 or 5; we may therefore (181) ascertain *the prime factors of the denominator,* which will determine *whether the decimal equivalent will be finite or infinite.* Hence, the possibility or impossibility of expressing in decimals the value of a vulgar fraction, depends *entirely upon its denominator;* for, if this contains *any prime factor other than* 2 *or* 5, it is impossible; if not, it is possible. Thus, we may have, in decimals, the exact value of $\frac{1}{16}$, $\frac{13}{32}$, $\frac{9}{64}$, $\frac{11}{20}$, $\frac{13}{25}$, $\frac{29}{50}$, $\frac{111}{125}$, $\frac{231}{400}$, &c., but not of $\frac{1}{3}$, $\frac{1}{6}$, $\frac{7}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, $\frac{19}{21}$, $\frac{37}{55}$, &c.

257. When the exact value of a fraction cannot be obtained in decimals, we can, by the following examination of its denominator, find *how many figures its decimal period will contain, and where that period will begin.*

Before the fractional equivalent of a repeating decimal is reduced to lower terms, its denominator (254) consists of *as many nines as there are places in the period, terminated by as many ciphers as there are places between the period and the comma.* Wherefore, as any of the numbers 9, 99, 999, &c. is prime to 10, if the denominator of the given fraction terminates with ciphers, there must be, in its decimal equivalent, *as many places between the comma and the period as there are terminating ciphers.* These ciphers may therefore be suppressed and counted as so many places. Now, as the denominator of the period is prime to 2 or 5, and as either of these is only once factor in 10, neither 2 nor 5 could be factor in the denominator of the given fraction, after the suppression of its terminating ciphers, unless 10 were as often factor in the denominator of the fractional equivalent first found, after the suppression of the same number of its terminating ciphers. Therefore, the remaining part of the given denominator, after the suppression of its terminating ciphers, may be divided by 2 and by 5 as often as possible; and the number of divisors employed will show the remaining number of times that 10 is factor in the fractional equivalent, found as in art. 254; consequently, the number of ciphers suppressed, if any, together with the number of divisors, will show *the number of times that* 10 *is factor in the denominator of the first fractional equivalent;* that is, the number of ciphers in this denominator, and finally (254) *the number of places between the comma and the period.*

Again, if we divide a unit by any number prime to 10, the quotient figures will repeat, *as soon as the remainder is* 1; therefore, as any of the numbers 9, 99, 999, &c. is 1 less than the corresponding one in the *series* (succession) 10, 100, 1000 &c., if we divide a series of nines by the number prime to 10, the quotient will be exact, *when we have reached the limit of the period.* But the decimal value of a fraction is the same

(165) in whatever terms it is expressed. Therefore, in dividing a series of nines by the remaining quotient, after having exhausted the factors 10, 5, and 2, found in the denominator of the given fraction, the division will be exact, when we reach the limit of the period; consequently, the number of nines divided will show *the number of figures in that period.*

To find whether $\frac{1039}{1040}$ is finite or infinite, and if infinite, of how many places of figures its period consists, as well as where it begins, we proceed thus:

```
2) 104,0
   ------
 2) 52
   ----
 2) 26
   ----
     13) 99 (76923
             89
             119
               29
                39
                **
```

Having found a factor 13, which is prime to 10, the decimal will be infinite, (256.) Again, as we have suppressed one cipher and employed three divisors, there will be four places between the comma and the period, and hence the period will begin at the fifth place. Finally, as we have employed six nines, in dividing the series to find an exact quotient, there will be six decimals in the period.

Proof. The value of $\frac{1039}{1040}$ in decimals is ,9990$\dot{3}$8461$\dot{5}$.

For practice, the scholar may examine the following fractions, and prove as above:

$\frac{13}{140}$, $\frac{17}{650}$, $\frac{2}{1375}$, and $\frac{2887}{3700}$.

258. A unit, divided by any of the numbers 9, 99, 999, &c., can give no other figure (253) in the quotient or remainder. Therefore, if we except the case of $\frac{9}{9} = 1$, any figure whatever divided by 9, 99, 999, &c., will always give *the same figure in the quotient and remainder.* Now, if by 9, we divide a figure in the unit place, which is less than 9, as we must first reduce it to tenths, the first quotient figure will be tenths, and the repeater will, of course, begin at the comma. If we divide any figure whatever by 99, as we must first reduce it to hundredths, the first quotient figure will be hundredths; and, of course, we shall have 0 in the place of

tenths: but, as the remainder is the same figure, which, when reduced to the next lower order, does not contain 99, the next quotient figure will be a cipher; consequently, in this case also, the period begins at the comma. In the same manner we might show that, in dividing by any other number of nines, *the period will begin at the comma.*

259. Excepting an equal number of nines, if by 9, 99, 999, &c., we divide any number of figures, not exceeding the number of nines divided by, these figures, in the order in which they are given, will be the only figures in the quotient and remainder; and *the period will begin at the comma and consist of as many places as there are nines in the divisor.*

For example, if we would divide 678 by 9999, we may consider the division of each figure separately, thus: As any of the numbers 10, 100, 1000, &c., is greater than the corresponding one of the numbers 9, 99, &c., if any figure has as many ciphers on the right as there are nines in the divisor, it will contain that divisor. Now, the figure 6, which is 600 units, must be reduced to 60000 hundredths, before it will contain 9999; consequently, the first figure 6, in the quotient and remainder must be 6 hundredths. The next figure 7, being 70 units, must, before we can divide, be reduced to 70000 thousandths; consequently, the figure 7 will take its place as a number of thousandths, in both quotient and remainder, on the right of the 6 hundredths already written in each. The last figure 8, being units, we reduce to 80000 ten-thousandths; consequently, the figure 8, as quotient and remainder, takes its place as 8 ten-thousandths, on the right of the 7 thousandths already written. Now, as the first figure 6 is hundredths, there must be one cipher between it and the comma; the quotient found is, therefore, ,0678, which begins at the comma.

When the number of figures in the divisor and dividend is the same, as we must, before we can divide, reduce the latter to tenths, the first quotient figure will be tenths: the period, therefore, in this case, evidently begins at the comma. Also, from the above, it is evident that, when the number of figures in both is not the same, the difference will show the number of ciphers between the first significant figure of the period and the comma. Wherefore, *the period always begins at the comma, and contains a number of figures equal to the number of nines in the divisor.*

260. A proper fraction, the denominator of which is prime

to 10, is equal (257) to *a fraction, the denominator of which is a number of nines, equal to the number of figures in its decimal period.* Now, each of these two fractions must give the same decimal quotient: but the numerator does not contain more figures than the denominator; and we have seen above that, in this case, the quotient will be a repeating decimal, beginning at the comma. Wherefore, when the denominator of a fraction is prime to 10, *its decimal period begins at the comma.*

For practice, the student may find the periodical decimal equivalents of the following fractions, performing the divisions as in Art. 243; namely, of

$$\tfrac{1}{7},\ \tfrac{1}{13},\ \tfrac{3}{17},\ \text{and}\ \tfrac{1}{71}.$$

Also, if he wishes to obtain the fractions again from the decimal periods, the last of which will consist of 35 figures, he may *find* (254) *the fractional equivalent of each, and divide both terms by their greatest common measure.*

261. If we repeat a decimal period any number of times, it will assume for a new denominator, *the number of nines in the denominator of its fractional equivalent, repeated the same number of times.* Thus, if we repeat the period $,\dot{0}67\dot{8}$ three times, the denominator of the fractional equivalent will be 999999999999: and this is evident, because the new numerator ,067806780678 is the product of $,000100010001\times 678$, and the new denominator the product of $,000100010001\times 9999$; and because, as we have seen, (259,) any number of figures, divided by an equal number of nines, *will repeat in their exact order, beginning at the comma.* Hence, also, a single repeater may be repeated any number of times, and will take, for a denominator, as many nines as there are places on on the right of the comma. Thus:

$$,\dot{6} = ,\dot{6}\dot{6} = ,\dot{6}6\dot{6} = ,\dot{6}66\dot{6},\ \text{\&c., or}\ \tfrac{6}{9} = \tfrac{66}{99} = \tfrac{666}{999} = \tfrac{6666}{9999},\ \text{\&c.}$$

262. Also, either of the above may (254) be represented in mixed form, thus:

$$,0\dot{6}78\dot{0};\ ,06\dot{7}80\dot{6};\ ,067\dot{8}06\dot{7};\ ,0678\dot{0}67\dot{8},\ \text{\&c.; and}$$

$$,6\dot{6};\ ,66\dot{6};\ ,6666\dot{6}666\dot{6},\ \text{\&c.}$$

Here we must observe, that although any period may be so expressed as to have any number of figures at pleasure between the comma and the period, it is only with a single repeater that we can make the period to consist of as many figures as we please; for example, if, in the expression $,\dot{0}67\dot{8}$, we leave out one or more figures, it is plain that, as these do not repeat, the value will be changed. Also, if we make the period to consist of more than four figures, unless the number is a multiple of four, the figures will not follow each other in the same order, and hence the order of their units will be changed. Therefore, we can only increase the number of figures in a period consisting of different figures, *by making the number of figures in the new period a multiple of the number in the given one.*

263. When the points which designate the period, which we call *the extremes*, are, in several decimal numbers, equidistant, that is, when the period in each begins at the same place, and has the same number of figures, *these numbers are similar*, seeing that their fractional equivalents must (254) have a common denominator. When the extremes are not equidistant, *the numbers are dissimilar*, because their fractional equivalents have different denominators.

If we have several dissimilar repeating decimals, which we wish to render similar, it is evident that the number of figures in some of the periods must be increased; but this (262) requires that the number of figures in the new period shall be *a multiple of the number in the given one:* if, therefore, we wish to find a common period for all the given numbers, which we must do before they can be similar, it is plain that the number of figures in this new period must be *a multiple of the number in each:* consequently, we take *the least common multiple* of the number in each, as the most convenient.

Again, if some of the decimal figures are mixed, we must be careful *to extend the definite figures*, (254;) namely, those between the comma and the first figure of the repeating period, *so that the common period may commence at the place next, on the right, to that of the lowest definite figure in any of the given numbers.* Thus, if we have

$$,\dot{6};\ ,0\dot{6};\ ,\dot{6}\dot{3}9;\ ,095\dot{2}\dot{7};\ \text{and}\ ,47\dot{6}2\dot{5},$$

which we would render similar, we count the number of figures in each period, and find respectively 1, 1, 2, 2, 3, the least common multiple of which is 6: wherefore, the common period must consist of 6 figures, and begin at the fourth place, the greatest number of definite figures being 3; that is to say,

,095. The numbers, when rendered similar, will, therefore, stand thus:

$$\begin{array}{r} ,666\dot{6}6666\dot{6}\\ ,066666666\\ ,639393939\\ ,095272727\\ ,476256256 \end{array}$$

Addition of Repeating Decimals.

264. As we cannot add units of different orders, to add repeating decimals, we first *render them similar*. The period in each is then *the numerator of a fraction*, and the common denominator (261) *as many nines as there are places in the common period*. Wherefore, the numbers being placed under each other, we find the sum of the periods, as in whole numbers; and, if the sum does not contain more figures than the common period,—that is, if the sum of the left-hand column of the numbers within the period does not exceed 9—*we write it underneath;* and, having marked the extremes of the total sum, we continue the addition as usual. But, if the sum of the left-hand column exceeds 9, *we not only carry the left-hand figure of this sum to the next column on the left, but we also add this same figure to the right-hand figure of the sum total of the periods.* The reason of this is, that this left-hand figure has as many places on the right of it, as there are nines in the denominator of the period; and, if we divide its value by the common denominator, which shows the order of its units, we shall have the same figure (258) for both quotient and remainder. But the quotient is evidently a number of units of the next higher decimal order, and the remainder, being a number of units of the order expressed by the common denominator, has its true value in being added to the unit figure of the period.

For example, if we would add the numbers given in the preceding article, having prepared and placed them as directed, we proceed thus:

$$\begin{array}{r} ,666\dot{6}6666\dot{6}\\ ,066666666\\ ,639393939\\ ,095272727\\ ,476256256\\ \hline 1,944256254\\ 22 \end{array}$$

the addition being performed, from beginning to end, as in whole numbers, with the sole exception, that having found the sum 22 of the left-hand column of the common period, we write its left-hand figure 2 under the unit figure 4 of the period, to remind us that it must be added to that figure. The sum of the given numbers is, therefore,

$$1{,}944\dot{2}5625\dot{6} \text{ or } 1{,}944\dot{2}5\dot{6}$$

The scholar may prove the work by finding (254) the fractional equivalents, and adding them, as usual in vulgar fractions, thus:

$$\tfrac{6}{9}+\tfrac{6}{90}+\tfrac{633}{990}+\tfrac{9432}{99000}+\tfrac{47578}{99900}=\tfrac{21365432}{10989000}=1{,}944\dot{2}5\dot{6}$$

If definite or finite decimals be found amongst the numbers to be added, it is plain that we must extend the definite figures of the mixed and repeating decimals, so that the common period may begin *at the next place on the right of the lowest order of definite figures* in any of the given numbers.

Examples.

1. $5{,}237+65{,}\dot{3}+16{,}78\dot{9}+750{,}0\dot{3}\dot{7}+{,}\dot{9}28\dot{6}=838{,}326\dot{3}99\dot{9}$
2. ${,}00\dot{7}\dot{5}+32{,}\dot{7}+113{,}\dot{0}0365\dot{2}+2{,}\dot{9}8\dot{7}+{,}2359=$
   $149{,}0128\dot{9}3526\dot{9}$
3. $25{,}\dot{8}+9{,}541+2{,}00\dot{7}+{,}6\dot{2}5\dot{8}+97{,}\dot{9}\dot{7}=136{,}043\dot{2}9047\dot{2}$

Subtraction of Repeating Decimals.

265. As we cannot subtract units of different orders, we render the numbers similar, as in addition, and place them so that the commas may be under each other. Then, if the number which constitutes the period in the lower number, be not greater than that of the period above it, as we have merely to diminish the upper period by a number of units of the same kind, *the subtraction is performed throughout as in integers, except the marking of the period in the remainder.*

But, if the number in the lower period be the greater, we consider that, if to the upper period we add a unit of the next higher decimal order, this unit is equal to the common denominator, and in adding this, which is a number of nines equal to the number of figures in the period, to the upper period, —that is to say, the alt minus 1,—the unit figure of this period will be one unit less, the other figures of the period will not be altered, and there will be a unit projected one place towards the left, which is ten with regard to the left-hand figure of this upper period. Now, in subtracting in the ordinary way, as the left-hand figure of the lower period, with what may be to carry, is supposed greater than the one above it, we shall have to add ten to this upper figure, and carry one to the next lower figure on the left: now, this carrying of one to the next lower figure on the left, is just what we should have to do in consequence of having added a unit of this order to the upper number. Wherefore, *we suppose the unit figure of the upper period to be one unit less;* the operation is, in every other respect, *the same as in whole numbers, except the marking of the extremes in the remainder.*

For example, to subtract $3{,}23\dot{5}\dot{7}$ from $8{,}0\dot{2}3\dot{5}$, we render the numbers similar, and place them thus:

$$\begin{array}{r} 8{,}02\dot{3}52352\dot{2} \\ 3{,}23575757 \\ \hline 4{,}78\dot{7}7659\dot{4} \end{array}$$

Then, as the number in the lower period is greater than that in the upper, instead of saying 7 from 12, we say, 7 from 11, *four*. In every other respect, we subtract as in whole numbers, except that in the remainder we mark the extremes of the period.

To prove the correctness of the above result, as well as of those in the following examples, the scholar may find the fractional equivalents, subtract as in vulgar fractions, and then reduce the fractional remainder to decimals. In dividing, he may proceed as in Art. 243, and, omitting the ciphers on the right of the divisor, continue the operation *till the quotient figures begin to repeat.*

Examples.

1. $5{,}\dot{6} - 2{,}6\dot{3}2\dot{1} = 3{,}0\dot{3}4\dot{5}$
2. $23{,}0\dot{0}\dot{5} - 7{,}35\dot{4}9\dot{2} = 15{,}65\dot{0}1255\dot{8}$
3. $42{,}00\dot{3}\dot{6} - {,}976\dot{8}\dot{4} = 41{,}026\dot{7}\dot{8}$
4. $378{,}2\dot{0}58\dot{3} - 67{,}3759\dot{8} = 310{,}8298\dot{4}16\dot{9}$

Multiplication of Repeating Decimals.

When the repeating decimal is found in the multiplicand only, and the multiplier is a single figure.

266. To multiply a number, is to add it to itself a certain number of times; we may, therefore, suppose the period of the repeating decimal written under itself a certain number of times, and the addition performed as in Art. 264; that is, if the sum or product of the left-hand order of the period does not exceed 9, *the whole work is performed as in whole numbers, except the marking of the extremes.* But if the sum or product of the highest order of the period exceeds 9, *we not only carry its tens to the next place on the left, but add them as units to the right-hand figure of the period in the product.*

If the period consists of a single figure, as this is a number of *ninths*, we divide its product by 9, setting down the remainder, which we mark as a repeater: then, carrying the quotient, we proceed as usual. For example, to multiply $6{,}3\dot{5}$ by 6,

$$\begin{array}{r} 6{,}3\dot{5} \\ 6 \\ \hline 38{,}1\dot{3} \end{array}$$

we first say, 6 times 5 is 30. Now this is $\frac{30}{9}$; and as the division gives the same figure for quotient and remainder, we write 3 and carry 3, after which we proceed as with ordinary decimals.

We may prove the work thus:

$$6{,}3\dot{5} \times 10 = 63\tfrac{5}{9} = \tfrac{572}{9} : \tfrac{572}{9} \div 10 = \tfrac{572}{90}$$

$$\frac{572}{\cancel{9}\cancel{0}} \times \frac{\cancel{6}}{1} = \frac{572}{15} = 38{,}1\dot{3} \text{ as before.}$$

Again, if we have $5{,}75\dot{8}$ to multiply by 7:

$$\begin{array}{r} 5{,}75\dot{8} \\ 7 \\ \hline 40{,}31\dot{2} \end{array}$$

we say 7 times 8 is 56; then to find the remainder, we cast out the nines, saying 5 and 6 is 11; 1 and 1 is 2; which we write and mark. Then adding the 9 cast out as a unit to the

5 nines contained in 5, the left-hand figure, we carry 6, and proceed as usual.

Proof. $575\frac{8}{9} = \frac{5183}{9}$: then $\frac{5183}{900} \times \frac{7}{1} = \frac{36281}{900} = \frac{362,81}{9}$ $= 40,31\dot{2}$ as before.

Examples.

1. $9,0\dot{8} \times 9 = 81,8$; $5,\dot{6} \times 3 = 17$; and $4,81\dot{3} \times 5 = 24,0\dot{6}$

2. $,1\dot{4} \times 9 = 1,3$; $,261\dot{7} \times 9 = 2,356$, and $3,\dot{3} \times 3 = 10$

The scholar will easily see, that when the multiplier is 9, it destroys the divisor 9, or denominator of the period; therefore, in the product, *the repeater vanishes.*

267. When the period contains more than one figure, we multiply as in integers; and, if the product of the last or left-hand figure of the period does not exceed 9, we write it underneath; and having marked the period, we continue the operation as usual. But if the product of this left-hand figure exceeds 9, we not only carry the left-hand figure of this product forward, to be added to the product of the next figure, but we also carry it backward, to be added to the unit figure of the period in the product. Now, as multiplication is a species of addition, the student is referred for the reason of this to art. 264.

For example, $23,\dot{1}6\dot{8} \times 4 = 92,\dot{6}7\dot{2}$; the multiplication being performed exactly as in whole numbers.

But, if we have $54,0\dot{9}8\dot{7}$ to multiply by 8, having multiplied the period and found 7896, as the 7 divided by 999 would give 7 for both quotient and remainder, we add 7 to the unit figure 6, which gives $\dot{9}0\dot{3}$ for the period; and carrying the quotient 7, we proceed as usual. Wherefore,

$$54,0\dot{9}8\dot{7} \times 8 = 432,7\dot{9}0\dot{3}$$

Proof. $54,0\dot{9}8\dot{7} \times 8 = \frac{540447}{9990} \times \frac{8}{1} = \frac{4323576}{9990} = 432,7\dot{9}0\dot{3}$

Examples.

1. $3,\dot{1}4\dot{2} \times 6 = 18,\dot{8}5\dot{2}$, and $14,\dot{0}2\dot{6} \times 9 = 126,\dot{2}3\dot{4}$

2. $4,07\dot{5}\dot{9} \times 8 = 32,60\dot{7}\dot{6}$, and $,0\dot{0}92\dot{5} \times 9 = ,0\dot{8}32\dot{5}$

3. $35,\dot{6}79\dot{8} \times 7 = 249,\dot{7}59\dot{0}$, and $,\dot{0}92\dot{8} \times 9 = ,\dot{8}36\dot{0}$

When the multiplier contains several figures.

268. When the period in the multiplicand contains but one figure, we multiply by each figure of the multiplier, as in art. 266; and having found the several products, *we continue the repeating figure in each to the unit place of the first partial product.* Then having found the sum of the right-hand column, which is evidently a number of ninths, we divide this sum by 9, setting down the remainder as ninths, and carrying the quotient to the next column, after which we add as usual.

For example, if we have $2{,}35\dot{4}$ to multiply by 125, having placed the numbers

$$\begin{array}{r} 2{,}35\dot{4} \\ 125 \\ \hline 11772 \\ 282533 \\ \hline 294{,}30\dot{5} \end{array}$$

and found the two partial products, in multiplying by 5 and by 12, as in art. 266, we repeat the figure 3 of the second product, so that it may stand under the repeater of the first; and having added the first column, we mark the sum 5 as a repeater; for it is evident that as both figures 3 and 2 repeat, their sum must also repeat.

269. A mixed decimal, as well as any other, may be multiplied by 10, 100, 1000, &c., *by removing the comma, as usual, towards the right;* but if it should extend into, or beyond the period, care must be taken *to repeat the period till there are as many figures on the right of the comma as are contained in the period.* If, in proceeding thus, the order of the figures in the period on the right of the comma appears to be changed, this is of no consequence; because they still repeat, in regular order, from the place where they begin, to infinity; and because *the ratio of increase and diminution is, in integral and decimal numbers, the same.*

Thus: $2{,}35\dot{4}\times1000=2354{,}\dot{4}$; also, $2{,}\dot{3}5\dot{4}\times100=235{,}\dot{4}3\dot{5}$

As the number 125 is the eighth part of 1000, if we divide $2354{,}\dot{4}$ by 8, we shall have the product of $2{,}35\dot{4}\times125$. Thus:

$$\begin{array}{r} 8)\underline{2354{,}444,\ \&c.} \\ 294{,}30\dot{5} \end{array}$$

where we repeat 4 in the dividend, till we find that the quotient figure 5 will repeat, which we mark accordingly. This may serve as proof of the example in art. 268.

To multiply $6{,}56\dot{7}$ by 25, we place the numbers and multiply as before.

$$\begin{array}{r} 6{,}56\dot{7} \\ \underline{25} \\ 32838 \\ \underline{131355} \\ 164{,}19\dot{4} \end{array}$$

But as the sum of 5 and 8, which is 13, exceeds 9, we divide by 9; or rather, we say 1 and three is 4, which we write as a repeater underneath, and carrying the quotient 1, we proceed as usual.

$$\begin{array}{r} 4\underline{)\,656{,}777,\ \&c.} \\ 164{,}19\dot{4} \end{array}$$

This last operation, which may serve as proof of the preceding, consists in multiplying by 100, and dividing by 4, because 25 is the fourth part of 100.

Examples.

1. $93{,}548\dot{5} \times 875 = 81854{,}986\dot{1}$
2. $275{,}053\dot{2} \times 927 = 254974{,}337$
3. $,584\dot{7} \times 6{,}578 = 3{,}846668\dot{2}$
4. $,01879\dot{8} \times 2{,}9479 = ,055417244\dot{5}$

270. When there are several figures in the period of the multiplicand, we multiply by each figure of the multiplier, as in art. 267; and, having found the several partial products, we continue the repeating figures of each, in their proper order, to the right-hand or lowest place of the first partial product. Then, taking as many of the right-hand columns as there are figures in the period of the multiplicand, we consider them as belonging to the common period, and add them as in art. 264. The reason for operating thus will appear plain, if we consider that *the period in the multiplicand is the numerator of a fraction, which does not change the denomination of its units in being*

multiplied by any number; the products, therefore, must all terminate at the same place as this period; and we must have just the same number of repeating figures in the sum of these products, or total product of the given numbers, as in the multiplicand.

For example, if we have $65{,}\dot{4}3\dot{7}$ to multiply by 3462:

$$\begin{array}{r} 65{,}\dot{4}3\dot{7} \\ \underline{3462} \\ 130874 \\ 3926246 \\ 26174974 \\ \underline{196312312} \\ 226544{,}\dot{4}0\dot{8} \end{array}$$

As the product of the period by 2 does not exceed 999, the first product is found as in whole numbers; its figures 874 showing the limit of the period. The second product, 392624, is first found as in art. 267. If this stood alone, as the product of the multiplicand by 6, its period would be $\dot{6}2\dot{4}$; but as the multiplier is 6 tens, this product must be ten times greater. Now, to multiply $392{,}\dot{6}2\dot{4}$ by 10, we remove the comma one place towards the right; and, as this enters the period, we must (269) repeat the figure 6, which it passes, by placing it on the right of 4. The second product is therefore $3926{,}\dot{2}4\dot{6}$. As the next multiplier is 400, we must, in the third product, suppose the comma to be removed two places, and repeat the figures 74 of its period. As the last multiplier is 3000, we must, in the product, repeat the whole period 312. Thus, *the products will always terminate at the right-hand place of the first product.*

The work may be proved by multiplying $65\frac{437}{999}$ by 3462, and reducing the product to decimals.

If the multiplier had been 34,62, the product would have been $2265{,}44\dot{4}0\dot{8}$.

Examples.

1. $5{,}\dot{1}7\dot{9} \times 2222 = 11508{,}\dot{1}3\dot{6}$
2. $643{,}\dot{3}46\dot{2} \times 59{,}4 = 38214{,}7\dot{6}63\dot{3}$
3. $7{,}3\dot{9}574\dot{8} \times 2731 = 20197{,}7\dot{9}040\dot{2}$

271. When there are repeating decimals in both factors, we may *find the fractional equivalent of the multiplier; multiply by its numerator* as in the preceding article, *and divide the product by the denominator,* taking care to repeat the period in the number we are dividing, till we obtain the quotient to a sufficient degree of exactness, or till its figures repeat, as may be thought proper.

Or, we may *find the equivalents of both factors, multiply them as vulgar fractions, and reduce the fractional product to decimals.* The scholar may operate by both methods, and see which he prefers; also, the one will be a proof of the other. For example, if we have $81{,}6\dot{2}7\dot{5}$ to multiply by $,3\dot{5}\dot{8}$, we may operate thus: $,3\dot{5}\dot{8} \times 10 = 3\frac{58}{99} = \frac{355}{99}$, and $\frac{355}{99} \div 10 = \frac{355}{990}$. Now, though this fraction might be reduced, it may be as well to let it remain as it is; then,

$$\begin{array}{r}
81{,}6\dot{2}7\dot{5} \\
355 \\
\hline
4081376 \\
285696346 \\
\hline
9\,)\,2897{,}77\dot{7}2\dot{2}722722 \\
\hline
11\,)\,321{,}97524746969 \\
\hline
29{,}27047704269+
\end{array}$$

having multiplied by 5, we multiply the first product by 7, for 35. To divide by 990, we remove the comma one place to the left, which divides by 10, and then divide by 9 and by 11, which are the factors of the remaining number 99.

Or thus: $,3\dot{5}\dot{8} = \frac{355}{990} = \frac{71}{198}$, and $81{,}6\dot{2}7\dot{5} = \frac{815459}{9990}$: then, $\frac{815459}{9990} \times \frac{71}{198} = \frac{57897589}{1978020} = 29{,}27047704269+$, as before.

Examples.

1 $,\dot{2}37\dot{6} \times ,\dot{3}\dot{2} = ,\dot{0}76\dot{8}$

2. $55{,}\dot{7}64\dot{5} \times ,\dot{3}56\dot{4} = 19{,}8764826977747279+$

3. $4{,}\dot{9}362\dot{5} \times ,\dot{7}62\dot{5} = 3{,}7642741913+$

4. $3{,}0\dot{5} \times 71{,}\dot{6}\dot{5} \times 4{,}\dot{6}3\dot{2} = 1014{,}3177745646+$

Division of Repeating Decimals.

272. *Find the fractional equivalents of the given numbers, divide as in vulgar fractions, being careful to cancel as much as possible, and reduce the quotient to decimals.*

If we have $9,\dot{2}67\dot{5}$ to divide by $2,\dot{5}\dot{9}$, we proceed thus:

$$9,\dot{2}67\dot{5} \div 2,\dot{5}\dot{9} = \tfrac{92666}{9999} \div \tfrac{257}{99} : \text{ then}$$

$$\frac{92666}{\cancel{9999}} \times \frac{\cancel{99}}{257} = \frac{92666}{101 \times 257} = \frac{92666}{25957} = 3,569981 +$$

Examples.

1. $63,\dot{8} \div 12,\dot{7} = 5$
2. $289,\dot{0}59\dot{7} \div ,\dot{3}64\dot{7} = 798$
3. $53,\dot{2}9645\dot{7} \div ,\dot{3}7\dot{2} = 143,12677644 +$

Or, *find the fractional equivalent of the divisor only; multiply the dividend by its denominator, and divide the product by the numerator.* By this last method the scholar may prove any of the above examples. He will also see that it is, in many cases, the most convenient.

273. We obtain a convenient method of multiplying or dividing by 5, 5^2, 5^3, 5^4, &c., *by finding the decimal value of a unit divided successively by each of those numbers*, as in the following table:

A unit divided by 5 = ,2
" " 25 = ,04
" " 125 = ,008
" " 625 = ,0016
" " 3125 = ,00032
&c., &c.

Now, if we divide any number by 5, it is evident that for every unit in it we shall have 2, which is the same as to multiply the number by ,2. Wherefore, to divide a number by any of the numbers in the left-hand column of the above table, we have only to *multiply by the corresponding one in the right-hand column.*

Thus, to divide 76354 by 625, we have

76354 × ,0016 = 122,1664, which is easily performed without even placing the numbers under each other.

Again, the dividend 1, divided by the quotient ,2, must (60) give the divisor 5; consequently, any number divided by ,2, will give 5 times that number. Wherefore, to multiply by any number in the left-hand column of the table, we have only to *divide by the corresponding number on the right.* Thus, to multiply 56479 by 3125, we have 56479 ÷ ,00032; which we perform by rendering the number of decimals equal in each, and suppressing the comma, thus:

$$\begin{array}{r} 4)\underline{5647900000} \\ 8)\underline{1411975000} \\ 176496875 \end{array} \qquad \begin{array}{r} \text{Proof.} \\ 56479 \\ \underline{3125} \\ 1411975 \\ \underline{1750849} \\ 176496875 \end{array}$$

Examples.

1. 584629 ÷ 5 = 116925,8
2. 584629 × 5 = 2923145
3. 63792877 × 25 = 1594821925
4. 63792877 ÷ 25 = 2551715,08
5. 919287325 ÷ 125 = 7354298,6
6. 919287325 × 125 = 114910915625
7. 3587624953 × 625 = 2242265595625
8. 3587624953 ÷ 625 = 5740199,9248
9. 11358642791 ÷ 3125 = 3634765,69312
10. 11358642791 × 3125 = 35495758721875

The above examples may be proved by multiplying or dividing in the ordinary way.

BOOK IV.

TABLES OF MONEYS, WEIGHTS, AND MEASURES, WITH CALCULATIONS AND ILLUSTRATIONS—OF TIME AND THE REGULATION OF CLOCKS—ASTRONOMICAL AND GEOGRAPHICAL MEASURE—REDUCTION, ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF COMPOUND NUMBERS—PRACTICE AND CALCULATION BY COMPLEMENTS—DUODECIMALS, BOARD MEASURE, &c.

SECTION XIII.

TABLES OF MONEYS, WEIGHTS, AND MEASURES, WITH CALCULATIONS AND ILLUSTRATIONS.

274. A number, when not applied to any particular species of quantity, is called *abstract*. Thus, when we say 3, or 3 times, the number 3 is abstract. When applied to a particular species, as when we say 3 books, 20 bushels, &c., it is called a *concrete* number. See Art. 18.

Hitherto we have treated of numbers only as abstract; but we shall shortly consider their application to the measurement and valuation of quantities. We have already observed (3) that, to measure a quantity we must compare it with some known quantity of the same kind, which is called a unit. Now, as quantities differ in their nature and magnitude, the units or measures to which they are compared vary accordingly.

275. A number which is made up of units of different magnitudes is called *a compound number.*[*] Thus, 6 pounds 15 shillings and 6 pence is a compound number, because the pound differs from the shilling, and each of these differs from the penny.

The following Tables will show what relation, or ratio, the different units, by which we usually measure quantities, have to each other.

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[*] Perhaps, as some have suggested, it might more properly be called *a complex number.*

276. Accounts are kept in England and Ireland, as well as in Canada and several other British colonies,—and formerly were in the United States,—*in pounds, shillings, pence, and farthings;* but, though each of these denominations has always the same ratio to each of the others, the value of each in one country is by no means the same as in every other.

English or Sterling Money.

The pound sterling is marked £, the shilling *s.*, the penny *d.*, and sometimes the farthing *qr.*; these being the initial letters of the Latin words *libra, solidus, denarius,* and *quadrans,* which signify *pound, shilling, penny,* and *farthing* respectively.

4 farthings, *qr.*, make	1 penny,	*d.*
12 pence, or 48 *qrs.*	1 shilling,	*s.*
20 shillings, 240*d.*, or 960 *qrs.*	1 pound,	£
21 shillings	1 guinea.	
5 shillings	1 crown.	

Farthings are usually written *as fractions of a penny,* thus:

$\frac{1}{4}$*d.*	one farthing.
$\frac{1}{2}$*d.*	two farthings, or half-penny.
$\frac{3}{4}$*d.*	three farthings.

PENCE TABLE.					TABLE OF SHILLINGS.		
d.	*s.*	*d.*	*s.*	*d.*	*s.*	£	*s.*
12	1	20	1	8	20	1	0
24	2	30	2	6	30	1	10
36	3	40	3	4	40	2	0
48	4	50	4	2	50	2	10
60	5	60	5	0	60	3	0
72	6	70	5	10	70	3	10
84	7	80	6	8	80	4	0
96	8	90	7	6	90	4	10
108	9	100	8	4	100	5	0
120	10	110	9	2	120	6	0
132	11	120	10	0	150	7	10
144	12	130	10	10	200	10	0

277. In the United States accounts are kept in dollars and cents, and the coins issued by government are of gold, silver, and copper: a large proportion, however, of the actual specie currency is a mixture of Spanish, French, English, and other coins. The decimal division of the following Table was adopted by the Federal Government, on account of the great facility it affords in calculation:

Federal Money.

10 mills, *m.*, make..........	1 cent,	*ct.*
10 cents......................	1 dime,	*d.*
10 dimes, or 100 cents......	1 dollar,	$
10 dollars.....................	1 eagle,	*E.*

Mills are merely nominal.

GOLD COINS.		SILVER COINS.	
Double Eagle............	$20	Dollar..................	100 cts.
Eagle	$10	Half Dollar............	50 cts.
Half Eagle	$5	Quarter Dollar	25 cts.
Quarter Eagle...........	$2½	Dime...................	10 cts.
Three Dollar piece......	$3	Half Dime.............	5 cts.
Gold Dollar..............	$1	Tri-cent piece.........	3 cts.

The copper coins are cent and half-cent.

In 12345 mills, how many eagles? How many dollars? Dimes? Cents? Read the number in dollars, cents, and mills. What part of a cent is the figure 5? Read the number in dollars and cents.

278. The unit of each of the different denominations of Federal Money, from the highest to the lowest, being formed *in the same manner as the unit of any order in an abstract number,*—that is to say, *of ten units of the next inferior denomination,*—it is evident that the four cardinal* operations may be performed upon these, *as upon abstract numbers.*

To find the sum of $29,37 and $156,257, we place units of the same order under each other, and add them as abstract numbers, thus:

$$\begin{array}{r} 29{,}37 \\ \underline{156{,}257} \\ 185{,}627 \end{array}$$

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* Cardinal, *chief, principal.*

The lowest denomination being mills, we may consider the sum 185627 as representing a number of mills. Now, in dividing this sum by the number of mills in a unit of any of the other denominations, it is evident that we may read it in any or all of those denominations; but, as accounts are kept in dollars and cents, it is usual to consider the dollars as whole numbers, and the inferior denominations as decimals; therefore, as 1000 mills make a dollar, we divide by 1000, in separating, by a comma, three figures on the right, and read 185 dollars and 627 thousandths, or (as 10 *m.* make 1 *c.*) 185 dollars 62 cents and 7 mills.

Hence, we see that the calculation of Federal Money is *the same as that of abstract decimals.*

Examples.

1. Find the sum of \$5,16; \$29,457; \$347,20, and \$1,627.
Answer, \$383,444.
2. From \$1000,50 subtract \$597,237.
Answer, \$403,263.
3. Multiply \$35,637 by 1,25.
Answer, \$44,54625, or \$44,55—.
4. Multiply \$379,23 by 25. Answer, \$9480,75.
5. Divide \$35,84 by 6,25. Answer, \$5,7344.
6. Divide \$12125 by 31,25. Answer, \$388.

279. Gold, silver, jewels, and very costly articles are weighed *by Troy Weight,* the denominations of which are as follows:

Troy Weight.

24 grains, *gr.*, make......	1 pennyweight, *dwt.*
20 pennyweights..........	1 ounce, *oz.*
12 ounces..................	1 pound, *lb.*

We refer the student to what is said in Art. 4, from which he will understand that the grains here spoken of are wheat grains, taken from the middle of a full-grown ear.

280. By the following weight apothecaries mix their medicines, but buy and sell by Avoirdupois:

Apothecaries' Weight.

20 grains, *gr.*, make......	1 scruple, ℈
3 scruples..................	1 drachm, ʒ
8 drachms..................	1 ounce, ℥
12 ounces..................	1 pound, ℔

Apothecaries' Wine Measure.

1 minim.
60 = 1 fluidrachm = ,2254 *c. in.*
480 = 8 = 1 *fl. oz.* = 1,8047 "
7680 = 128 = 16 = 1 *pt.* = 28,875 "
61440 = 1024 = 128 = 8 = 1 *gal.* = 231 cubic inches.

Imperial Measure.

Gr. Troy.
1 minim = ,91
60 = 1 *fl. dr.* = 54,7
480 = 8 = 1 *fl. oz.* = 437,5 = 1 *oz.*
9600 = 160 = 20 = 1 *pt.* = 8750 = 1,25 *lb.*
76800 = 1280 = 160 = 8 *pts.* = 1 *gal.* = 70000 = 10 *lb.*

Tea-cup = 4 fluid ounces.
Wine-glass = 2 "
Table-spoon = $\frac{1}{2}$ "
Tea-spoon = 1 fluidrachm = $\frac{1}{8}$ fl. oz.

281. By the following weight are weighed things of a gross and drossy nature, groceries, chandlers' wares, and metals, except gold and silver. Its name, *Avoirdupois*, is derived from the French words *avoir*, to have, *du*, of the, and *poids*, weight; that is, to have great weight.

Avoirdupois Weight.

16 drachms, *dr.*, make	1 ounce, *oz.*
16 ounces	1 pound, *lb.*
28 pounds.........	1 quarter, *qr.*
4 quarters, or 112 *lbs*..........	1 hundredweight, *cwt.*
[*]20 hundredweight, or 2240 *lbs.*	1 ton, *T.*

A hundredweight is sometimes called *a quintal.*

The pound Avoirdupois consists of 7000 grains Troy, and is to the pound Troy very nearly as 17 to 14.

282. We have already noticed that 3 barley-corns, placed lengthwise in contact, make 1 inch, and 12 inches, 1 foot. The foot, thus constituted, is, however, differently divided to suit different kinds of measurement. In Carpentry, it is divided *into inches, and these into halves, quarters, eighths, and sixteenths.* In Civil Engineering, for the sake of the convenience of decimal calculation, it is divided *into tenths and*

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[*] By some a short ton is sometimes used, consisting of 2000 lbs, and a short cwt., consisting of 100 lbs.

hundredths; and, in philosophical investigations, it is frequently divided *into inches and lines,* as in the following Table:

Long Measure.

10 lines Eng., or 12 lines Fr., *l.*, make...	1 inch, *in.*
12 inches....................................	1 foot, *ft.*
3 feet.......................................	1 yard, *yd.*
6 feet.......................................	1 fathom, *fath.*
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet......................	1 rod or pole, *P.*
40 poles, or 220 yards......................	1 furlong, *fur.*
8 furlongs, or 1760 yards................	1 mile, *M.*
3 miles.......................................	1 league, *L.*

A *hand,* by which horses are measured, is 4 inches.

Though the league and fathom are chiefly confined *to maritime*[*] *affairs,* they are, by some nations, and particularly the French, often used to denote distances on land.

283. The scholar has seen (211) that a compound fraction is the product of two or more simple fractions. But (188) every fraction is a ratio; wherefore, substituting the word *ratio* for the word *fraction,* we say that *a compound ratio is the product of two or more simple ratios.* Hence, *every compound fraction is a compound ratio.*

A unit of any of the lower denominations of a compound number is *a fraction of that of the next higher,* and a compound fraction *of a unit of any order higher than the next.* For example, a farthing compared with a penny is $\frac{1}{4}$, which is a simple fraction. But, as the penny is $\frac{1}{12}$ of a shilling, the farthing, compared with the shilling, is $\frac{1}{4}$ of $\frac{1}{12}$, which is a compound fraction. Compared with a pound, it is $\frac{1}{4}$ of $\frac{1}{12}$ of $\frac{1}{20}$, which is also a compound fraction.

As $\frac{1}{4}$ is the ratio of a farthing to a penny, and $\frac{1}{12}$, the ratio of a penny to a shilling; and as $\frac{1}{48}$, the ratio of a farthing to a shilling, is formed by multiplying $\frac{1}{4}$ and $\frac{1}{12}$ together, the ratio of a farthing to a shilling is said to be compounded *of the ratio of a farthing to a penny, and of that of a penny to a shilling.* The ratio of a farthing to a pound is a ratio compounded of *the ratio of a farthing to a penny, of the ratio of a penny to a shilling, and of that of a shilling to a pound.* Thus, $\frac{1}{4}$ of $\frac{1}{12}$ of $\frac{1}{20} = \frac{1}{4} \times \frac{1}{12} \times \frac{1}{20} = \frac{1}{960}$; that is, $\frac{1}{960}$ is a ratio compounded of the ratios $\frac{1}{4}$, $\frac{1}{12}$, and $\frac{1}{20}$.

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[*] *Maritime,* belonging to the ocean.

The ratio of 1 inch to 1 mile is $\frac{1}{63360}$; of what ratios is it compounded?

284. By the following measure are measured cloths, laces, ribbons, &c.:

Cloth Measure.

$2\frac{1}{4}$ inches, *in.*, make	1 nail, *n.*
4 nails	1 quarter, *qr.*
4 quarters	1 yard, *yd.*
$2\frac{1}{2}$ quarters, or 10 nails	1 ell Hamburgh, *E. H.*
3 quarters	1 ell Flemish, *E. F.*
5 quarters	1 ell English, *E. E.*
6 quarters	1 ell French, *E. Fr.*

The ratio of 1 inch to 1 *E. Fr.* is $\frac{1}{54}$; of what seven ratios is it compounded? What is the value of each ratio?

285. The scholar has already been informed (110) that the space contained in any surface, called *its content*, is estimated *in squares*, and that the content of any right-angled figure of four sides is found by multiplying its length, taken in any measure, by its breadth, taken in the same measure. The units of the following Table are those chiefly used for measuring surfaces; and, as some of them refer solely to the measurement of land, the table is called

Land or Square Measure.

100 Eng. or 144 Fr. square lines, *s. l.*, make	1 square inch, *s. in.*
144 square inches	1 square foot, *s. f.*
9 square feet	1 square yd, *s. yd.*
$30\frac{1}{4}$ square yards, or $272\frac{1}{4}$ *s. f.*	1 pole, or perch, *p.*
40 poles in length, and one in breadth	1 rood, *r.*
4 roods, or 160 poles	1 acre, *a.*
4840 *s. yds.*, or 43560 *s. f.*	1 acre.
640 acres	1 square mile, *s. m.*

In the actual measurement of land an instrument is usually employed, called the

Surveyor's or Gunter's Chain.

7,92 inches make	1 link.
25 links	1 rod, pole, or perch.
100 links, 22 *yds.*, or 66 *ft*	1 chain = 4 rods.
80 chains	1 mile.
10 square chains—that is, 10 chains in length, and 1 in breadth—make	1 acre.
100000 square links	1 acre.

The length, 10 chains, is 1000 links, and the breadth, 1 chain, is 100 links, and 1000 × 100 = 100000, the content of an acre in square links.

286. To find the content of any right-angled piece of ground with Gunter's Chain, *multiply the length in links by the breadth in the same, and divide by the number of square links in an acre;* that is, *point off, in the product, five figures on the right for decimals.* The figures on the left of the comma will be *acres. Multiply the decimals by* 4, *pointing off five,* as before, and the figures on the left of the comma will be roods, or *quarters of an acre.* Again, *multiply the decimals by* 40, *pointing off as before,* and, on the left of the comma you have the *poles.*

For example, the length of a piece of ground is 30 chains, 57 links, and the breadth 20 ch. 23 links; what is its content?

$$
\begin{array}{r}
3057 \\
2023 \\
\hline
9171 \\
6114 \\
6114 \\
\hline
61,84311 \\
4 \\
\hline
3,37244 \\
40 \\
\hline
14,89760
\end{array}
$$

The content is, therefore, 61 *A.* 3 *R.* 15 *P.*, very nearly.

If you have not Gunter's Chain, *multiply the length in yards, by the breadth in yards, and divide the product by* 4840; *or,* for greater accuracy, *the length in feet by the breadth in feet, and divide the product by* 43560. In the above example the length being 2017,6 ft., and the breadth 1335,2 ft., I proceed thus:

$$2017,6 \times 1335,2 = 2693900 — . \quad \text{Then,}$$

$$\frac{2693900}{43560} = 61\,A.\ 3\,R.\ 15\,P., \text{ very nearly, as before.}$$

The ratio of a square Fr. line to a square mile is $\frac{1}{578086502400}$; of what ratios is it compounded? What is the value of each?

287. We have shown the application of the preceeding measure to surfaces, or figures which have only two dimensions—namely, length and breadth, (for, however irregular or various

their boundaries, they may all be reduced to a mean length and breadth;) the following is that which is applied to solid bodies, or magnitudes of three dimensions—that is, having *length, breadth, and thickness.* The scholar has already seen that the content of such is estimated *in cubes,* and is the product of *the length, breadth, and thickness, all taken in the same measure.*

Solid or Cubic Measure.

1000 Eng. or 1728 Fr. cubic lines, *c. l.,* make	1 cubic inch, *c. in.*
1728 cubic inches	1 cubic foot, *c. ft.*
27 cubic feet	1 cubic yard, *c. yd.*
40 feet of round, or 50 *ft.* of hewn timber	1 ton, *t.*
42 cubic feet	1 ton of shipping.
Firewood, 8 *ft.* long, 4 *ft.* broad, and 4 *ft.* high, or 128 cubic feet	1 cord, *c.*
16 *c. ft.* or $\frac{1}{8}$ of a cord, in the N.E. States,	1 foot of wood.

How many Eng. cubic lines in a cubic mile?

Ans. Two hundred and fifty-four quadrillions, three hundred and fifty-eight trillions, sixty-one billions, and fifty-six millions.

288. The following measure is applied to dry articles, such as corn, fruit, seed, roots, salt, sand, oysters, coal, &c. The principal unit of this measure is *the Winchester bushel,* which is *a round vessel* $18\frac{1}{2}$ *inches in diameter throughout, and 8 inches deep,* and contains $2150\frac{2}{5}+$ cubic inches. The dry gallon contains, of course, $268\frac{4}{5}$ cubic inches.

Dry Measure.

2 pints make	1 quart, *qt.*
4 quarts	1 gallon, *gal.*
2 gallons, or 8 quarts	1 peck, *pk.*
4 pecks, 8 *gals.,* or 32 *qts.*	1 bushel, *bush.*

As $\frac{1728}{2150}=\frac{4}{5}$, the cubic foot, 1728 cubic inches, is to the bushel, 2150 cubic inches, very nearly as 4 to 5. But the greater the magnitude of the unit, the less must be the number to make a certain quantity, and the less the magnitude, the greater the number; wherefore, the numbers are *inversely as the magnitudes;* that is, the magnitude of the foot being to that of the bushel as 4 to 5, the number of feet equivalent to a given number of bushels must be as 5 to 4. Therefore, 5 feet are nearly equivalent to 4 bushels; and hence we have the following rule, which will be found very useful in rural economy, commerce, and shipping:—

To reduce bushels to feet: To the number of bushels *add* $\frac{1}{4}$ *of itself.* To reduce feet to bushels: From the number of feet *subtract* $\frac{1}{5}$ *of itself.*

If we have 1728 for the given number of bushels, we shall, by the rule, find its equivalent in cubic feet to be 2160, which is too great by 10, or $\frac{1}{216}$ part of itself, the true equivalent being 2150. Therefore, in finding by the rule, the equivalent, in feet, of a given number of bushels, we must, if we wish it exact, *diminish it by* $\frac{1}{216}$ *of itself.*

Again, if we have 2150 for the given number of cubic feet, we shall, by the rule, find, for its equivalent in bushels, 1720. But this is too little by 8 bushels, or $\frac{1}{215}$ part of itself, the true equivalent being 1728. Therefore, in finding by the rule the equivalent in bushels, of a given number of cubic feet, we must, if we wish it exact, *increase it by* $\frac{1}{215}$ *of itself.*

289. We may however observe, that for any practical purpose, such as the construction of corn-cribs, caves, bins, granaries, pits, boxes, &c., for the preservation of corn, roots, or any other articles measured by Dry Measure, or for the calculation of the capacity of such cribs, granaries, &c., *the rule will be found sufficiently exact without the addition or subtraction of the fractional part.*

Suppose we would construct a crib to hold 1500 bushels of corn. As corn is put away in the ear, and as it requires, in measuring, to allow two bushels for one, and one of these well heaped, we shall allow $2\frac{1}{5}$ times the whole quantity of shelled bushels. We therefore say, $1500 \times 2\frac{1}{5} = 3300$; adding to this $\frac{1}{4}$ of itself, we have $3300 + 825 = 4125$, the content of the crib in cubic feet. Now as this is a quantity of 3 dimensions, one of which is arbitrary—seeing that the average width of a corn-crib is generally not more than 4 or 5 feet inside, that the air may pass freely—we find the factors of 4125, or inside dimensions of the crib thus: dividing by any convenient width, say 5 feet, we have $\frac{4125}{5} = 825$, for the area or content of the side. Then, if we wish the crib to be of a certain height, or length, we must *divide the area of the side by that height or that length,* and the quotient will be *the remaining factor;* that is *the length or height* accordingly. Thus, if we say 10 feet high, we have

$\frac{825}{10} = 82\frac{1}{2}$ feet, for the length. If we say 55 feet long, we have $\frac{825}{55} = 15$ feet, for the height of the crib.

If the crib be made only 4 feet wide at the bottom, it must be 6 feet wide at the top, because $\frac{1}{2}$ the sum of these will give the average width, 5 feet.

What must be the length of a crib to hold 3600 bushels of corn, the average width being 5 feet, and the height 12?

Answer, 165 feet.

What must be the height of a crib to hold 500 bushels of corn, the average width being 4 ft. 2 in., and the length 40 ft. 6 in?

Answer, 8 ft. $1\frac{7}{9}$ in.

290. If, of several factors, any one be diminished in a certain ratio, that is, by a certain part of itself, *the whole product of those factors will be diminished in the same ratio;* that is, *by the same part of itself.*

Let P represent the product of the undiminished factors, and A the factor we intend to diminish. Then PA is the total product. Now, if we diminish A by $\frac{1}{4}$ of itself; thus, $A - \frac{A}{4}$, the remainder $\frac{3A}{4}$, multiplied by P, will be $\frac{3PA}{4}$. But $PA - \frac{PA}{4} = \frac{4PA - PA}{4} = \frac{3PA}{4}$. Wherefore, the total product of PA is diminished in the same ratio as the factor A.

Having a granary 43 feet long, 10 ft. wide, and 10 ft. high, we wish to know how many bushels of grain it will hold. Now, as by the rule (228) we must, from the content of the granary in cubic feet, which is $43 \times 10 \times 10$, subtract $\frac{1}{5}$ of itself; we subtract from 10, which is of the factors, $\frac{1}{5}$ of itself, which gives $43 \times 10 \times 8 = 3440$ for the content in bushels. But this is too little by $\frac{1}{215}$ of itself. We therefore say, $\frac{3440}{215} = 16$, and $3440 + 16 = 3456$, the number of bushels required.

Proof. $\frac{43 \times 10 \times 10 \times 1728}{2150} = \frac{43 \times 1728}{21{,}5} = 2 \times 1728 =$ 3456 bush., as before.

Having 6912 bushels of grain, we wish to know to what depth it will cover a floor that is 43 ft. long and 25 ft. wide.

4) 6912
1728 one-fourth added.
216) 8640 c. ft., by the Rule $\frac{1}{216}$ too great.
40 is $\frac{1}{216}$, which we subtract.
8600 true content of grain in c. feet.

$43 \times 25 = \frac{4300}{4} = 1075$ area, or surface of floor.

$8600 \div 1075 = 8$ feet, the required depth.

Proof. $43 \times 20 \times 8 = 6880$, too little by $\frac{1}{215}$ of itself. $\frac{6880}{215} = 32$; then, $6880 + 32 = 6912$ bushels.

291. In the heaped bushel, used in measuring fruit, roots, &c., it is found that, according to pretty general usage, the heaped part is only *about $\frac{1}{5}$ of the capacity of the vessel*, which, as we shall shortly see, is by no means a large allowance. But custom is law, at least to the calculator; wherefore, applying to this our approximate rule, (228,) as the internal capacity of the bushel (called *the strick bushel*) compared with the cubic foot, is as 5 to 4, and the heaped part $\frac{1}{5}$ of 5, or 1, we have, for the ratio of the heaped bushel to the foot 6 to 4, or 3 to 2; hence the following rule :—

To reduce heaped bushels to cubic feet: *To the number of heaped bushels, add its half.* To reduce cubic feet to heaped bushels: *From the number of cubic feet, subtract its third part.*

To what length must a pit, 4 ft. wide and 3 ft. deep, be dug, in order to contain 3000 bushels of potatoes?

Ans. 375 feet.

How many bushels of coal can be stored in a cellar which is 16 ft. long, 10 ft. wide, and 9 ft. high? Ans. 960.

292. In England, a bushel of potatoes is used consisting of 80 lbs. weight Avoirdupois. This was probably based upon the following experiment: Let a tight half-bushel measure be filled with good-sized potatoes, by packing them with the hand; then, if water be poured into it amongst the potatoes, it will contain about 8 quarts before the water begins to run over, which proves that, when a half-bushel is thus filled, only one half its solid capacity is occupied by the potatoes; much less when, as usual, it is filled with the shovel.

If we measure 80 lbs. of potatoes, shovelling them on the half-bushel as long as they will conveniently lie, the measure will be thus filled 3 times. A bushel, therefore, as they are frequently measured, will only weigh about 53 lbs. Now, as a bushel of good wheat, which weighs from 62 to 64 lbs., is often lower in price than a bushel of potatoes, the student will see that it is important to estimate as well as calculate. In Maine the bushel of potatoes is 60 lbs, Avoirdupois.

293. By the following measure are measured brandy, spirits, cider, vinegar, molasses, oil, &c. Honey is sold by the pound Avoirdupois.

The gallon contains 231 cubic inches. Ten gallons make one anker.

Wine Measure.

4 gills, *gi.*, make	1 pint, *pt.*
2 pints	1 quart, *qt.*
4 quarts	1 gallon, *gal.*
31½ gallons	1 barrel, *bbl.* or *bar.*
42 gallons	1 tierce, *tier.*
63 gallons	1 hogshead, *hhd.*
84 gallons	1 puncheon, *pun.*
2 hogsheads, or 126 gallons	1 pipe, *p.* or butt, *b.*
2 pipes, or 252 gallons	1 tun, *T.*

The following measure is used for malt liquor only. The gallon contains 282 cubic inches.

Beer Measure.

2 pints, *pts.*, make	1 quart, *qt.*
4 quarts	1 gallon, *gal.*
9 gallons	1 firkin, *fir.*
2 firkins	1 kilderkin, *kil.*
2 kilderkins	1 barrel, *bar.*
3 kilderkins, or 1½ *bar.*	1 hogshead, *hhd.*
2 hogsheads	1 butt, *b.*
2 butts	1 tun, *T.*

SECTION XIV.

OF TIME, AND THE REGULATION OF CLOCKS—ASTRONO AND GEOGRAPHICAL MEASURE.

294. No subject that we can contemplate is of deeper interest, or more serious consequence to mankind, than time. The wise in all ages have testified that on the proper improvement of it depends our present as well as future happiness; and, though to some it may appear irrelevant to introduce in this place subjects of graver import than mere calculation, it may not be amiss to remind the student, in the words of Don José de Cadalso, (in a little satirical work entitled Los Eruditos á la

Violeta, published in honour of those who pretend to great knowledge with little study,) that the common object of all the sciences, and their utility to man, may be divided into two parts: one of which is, *to obtain a less imperfect knowledge of the Supreme Being*, by which the heart of man is disposed to render him more profound and sincere worship; the other is, *to render men more social;* for, by imparting to each other the productions of their understanding, they become united, as we may say, in spite of ocean and distance.

The word time, though of frequent recurrence in conversation, is by no means easy to define. Perhaps we may call it the measure of existence or the regular flow of duration.

Duration, if unlimited, is called *eternity:* time is therefore *a limited duration.*

We have already observed (4) that time is measured by the revolution of the earth about its axis, which is a sublimely regular motion. By this revolution, which is from west to east, the sun, the fixed stars, and other celestial objects, are made, apparently, to perform *a revolution round the earth from east to west.*

Besides the motion of the earth about its axis, *it continually revolves about the sun from west to east,* in an elliptical* orbit, at the mean or average distance of about 95 millions of miles. The period of this revolution is called *a tropical year.*

The mean diameter of the earth's orbit is therefore nearly 190, or, as we say, in round numbers, *nearly* 200 *millions of miles.*

This distance, however, compared with that of the nearest of the fixed stars, is *a mere point,* so very insignificant, that they are seen from any part of our orbit, *exactly in the same point of the heavens.*

From the observations of the most skilful astronomers, made with the finest instruments, it is concluded that Sirius, or the Dog Star, which, from its superior magnitude and brilliancy, has been considered the nearest of them all, cannot be at a less distance from the sun than *ten millions of millions of miles;* some have supposed that it is not nearer than 300 millions of millions; from which the student will infer, that with respect to the distance of these splendid orbs from each other, man, with all his boasted science, is reduced to *mere conjecture;* yet he must by no means imagine that the observations made upon the starry heavens with powerful instruments, have been made in vain. They have, on the contrary, been of incalculable advantage in establishing many important and useful facts, upon which are founded the regulations of the calculation of time; and, more especially, in perfecting the science of navigation. They have opened to our view, in an awe-inspiring manner, the immensity, beauty, and harmony of the works of the Almighty, profusely disseminated in the fearful depths of illimitable space, and thus, by showing man how limited is his sphere of action in this vast creation, assured him of *the insignificance and folly of human pride.*

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* *Elliptical,* somewhat oval.

In a clear frosty night, we think we see with the naked eye an innumerable multitude of stars; but the fact is that we cannot, under the most favorable circumstances, see more than *two or three thousand* above the horizon at the same time; but we are informed that Dr. Herschel saw 588 stars in the field of his telescope at once, and computed, in a very small portion of the celestial hemisphere 258000 stars.

To give the student a clearer idea of the distance of these orbs from each other, let us suppose the earth stationary at the nearest probable distance, and no doubt we might say possible distance, from Sirius, namely 10 millions of millions of miles; that the creation took place 6000 years ago, (the year consisting of 365¼ days;) and that a cannon ball, shot from Sirius at that time, had continued to move, with its initial velocity of 1200 feet per second, in a direct line towards the earth: we find that it would have travelled 43033090909$\frac{1}{11}$ miles, or not quite the two hundred and thirtieth part of the distance; and hence, that it would require, to perform the whole of its flight, upwards of *one million three hundred and eighty thousand years.*

Let us close this article with the sublime words of the Prophet Isaiah, c. 40, v. 26: "Lift up your eyes on high, and behold who hath created these things, that bringeth out their host by number: he calleth them all by names, by the greatness of his might, for that he is strong in power; not one faileth."

295. The extremities of the earth's axis—which points, with respect to the diurnal revolution, appear stationary—are called *the North and South poles.*

The equator is an imaginary line which passes round the earth and is equally distant from either pole.

A line drawn directly over the head of an observer, and pointing north and south, is called *a meridian line*, or *the meridian* of the place where he stands. This line, in its extension round the earth, passes *through the poles*, and cuts the equator *at right angles*.

When a star, in its apparent revolution, passes the meridian of any place, it is said to *culminate*, and the time which elapses between any two culminations of a fixed star, or one entire revolution of the earth about its axis, is called *a sidereal day*, from the Latin *sidus*, a star.

296. From the time that the sun's centre passes the meridian till it returns to the meridian again, is called *a solar day*.

As the sun is not in the centre of the earth's orbit,* the motion of the earth in its orbit is somewhat accelerated by a nearer approach to, and retarded by a recession from that luminary. From this, as well as other causes, the solar day is *not of uniform length*; it is, however, *the medium length of a solar day*, or average length of all the solar days in a year, which is the time shown by a well-regulated clock; that is to say, we divide the time of a *mean solar day* into 24 hours, such as shown by the clock. This is the time in common use, called by astronomers

Mean Solar, or Common Time.

60 seconds, *sec.* make	1 minute, *m.*
60 minutes	1 hour, *h.*
24 hours	1 day, *d.*
7 days	1 week, *w.*
2 weeks	1 fortnight.
4 weeks	1 month, *mo.*
13 months, 1 day, and 6 hours, or 365 days 6 hours	1 common or Julian year, *y.*

A labourer's month consists *of* 26 *working days*, and a mercantile month *of* 30 *days*.

The odd 6 hours of the Julian year are not reckoned till they amount to a day: a common year, therefore, consists *of* 365 *days*, and every fourth year, called bissextile or leap-year, *of* 366.

The year is also divided into 12 calendar months, which, in number of days, are irregular, as follows:

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* *Orbit*, the figure circumscribed by the revolution of a planet.

1st month...	January.........	31 days.
2d...........	February.......	28 "
3d...........	March..........	31 "
4th...........	April...........	30 "
5th...........	May.............	31 "
6th...........	June............	30 "
7th...........	July.............	31 "
8th...........	August..........	31 "
9th...........	September......	30 "
10th...........	October.........	31 "
11th...........	November.......	30 "
12th...........	December.......	31 "

When the year is exactly divisible by 4, it is leap-year, in which the second month (February) has 29 days.

297. The solar or tropical year is found, by observation, to consist of 365 d. 5 h. 48 m. 51 sec.

Now, as the motion of the earth in its orbit and that on its axis are both in the same direction, it is evident that the earth must perform *rather more than one entire revolution on its axis, while the sun, at a mean rate, appears to pass from any meridian to the same meridian again,* and that the successive increments* will, in one year, amount to *one entire revolution of the earth about its axis;* that is, the earth will, in one year, perform one more revolution on its axis than the number of apparent revolutions made by the sun in the same time. The ratio, therefore, of the sidereal day to the solar, being inversely as the number of each which composes the solar year, will be *nearly as* 365 *to* 366, or $\frac{365}{366}$, and, therefore, if we multiply 24 hours by 365 and divide by 366, the result will be the length of a sidereal day, in mean solar time, very nearly. Thus: $\frac{365 \times 24}{366} = 23$ h. 56 m. 4 sec. —, the length of a sidereal day, or the time in which the earth performs one revolution on its axis.

If we subtract the length of the sidereal from that of the solar day, we find that the difference is 3 m. 56 sec., or *very nearly* 4 *minutes.* Hence, a fixed star will culminate on any day *nearly* 4 *minutes, by the clock, sooner than on the preceeding day.* If, therefore, you stretch a meridian line over head, and with a plumb line set a post directly underneath, you may fix in the post a piece of tin with a very small hole in it, through which you may observe the culmination of any fixed star. A star of the first magnitude, and having a good elevation, is best; such as Aldebaran in Taurus, Bellatrix in Orion, or any other fixed star, which you can easily recognize. Then, if, by your clock or watch, the star culminates about 4 *minutes sooner each night than on the preceding*, the clock or watch is well regulated.

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* *Increments,* portions of increase.

If, by the clock, the star culminates more than 4 minutes sooner, *the clock goes too slow*, and *the excess of time above* 4 *minutes* is the time the clock has lost. If, on the contrary, the star does not culminate till the same hour as on the preceding evening, *the clock has gained* 4 *minutes;* and if it does not culminate till past the hour by the clock, 4 *minutes added to the time past the hour*, will show how much the clock has gained.

298. For the exact time of day, observe, through *a screen** *of some kind, to protect the eye*, the time when the sun's centre passes the meridian. This will be the mean solar noon, if the observation is taken *on or about the* 15*th of April, the* 15*th of June, the* 1*st of September, or the* 24*th of December.* If the observation is taken between those times, look at the Almanack for the equation of time: then, if the sun is fast of the clock, *subtract from the apparent noon;* if slow, *add to the apparent noon*, the equation of time, which will give the true time by the clock.

299. The true solar year not being equal to the year of 365 days 6 hours upon which Julius Cæsar established the leap-year, (the difference, 11 m. 9 sec., amounting, in about 130 years, to a whole day,) Pope Gregory XIII. ordered that every 100th year, which, according to the Julian method, would be bissextile, should be a common year of 365 days, except those centurial years, which, when we cut off two ciphers on the right, are divisible by 4, which were to remain bissextile. This is called the *Gregorian method*, or *new style*.

300. If the fixed stars could be seen in the daytime, the sun would, by the earth's daily motion in its orbit, from west to east, appear to progress among them in the same direction. The great circle or path described by the sun among the fixed

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* Very important. The omission would certainly injure the sight of the observer.

stars, in his apparent annual revolution, caused by the real revolution of the earth, is called by astronomers, *the ecliptic.* A broad circle of the heavens, about 8 degrees on each side of the ecliptic, is called *the zodiac,* from the Greek *zo-on,* an animal, because most of the twelve parts into which it was divided by the ancient Chaldeans or Egyptians were named after animals. The following are the divisions of the ecliptic or circle of the zodiac:

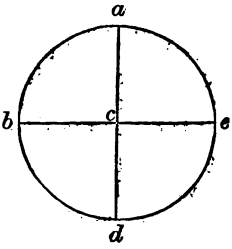
Astronomical Measure.

60 seconds (″) make...... 1 minute, ′.
60 minutes................ 1 degree, °.
30 degrees................ 1 sign, s.
12 signs, or 360°, the whole circle of the zodiac.

Astronomers apply the above measure in calculating the motions and angular distances of the planets and other celestial bodies.

301. A *circle* is *a round figure in a plane,* (104,) generated *by the revolution of a finite straight line in that plane, about one of its extremities, which remains fixed;* and this fixed point is called *the centre.* The line described by the motion of the other extremity is called *the circumference,* which is, of course, in all its parts, *equally distant from the centre;* and hence all straight lines drawn from the centre to the circumference *are equal.* These straight lines are called *radii* of the circle, and any one of them *a radius.*

If two straight lines cut one another at right angles, (103,) they may be considered as four straight lines which meet in one point, making equal angles with each other. Now, if round the point in which they meet, called *the point of intersection,* we describe a circle, *its circumference will be divided into* 4 *equal parts,* called *arcs.* Thus, in the circle *abde,* the angles at the point *c,* being right angles, are equal to one another, and the arcs *ae, ed, db,* and *ba,* upon which they stand, are also equal.



First, it is evident that a straight line drawn through the centre, from side to side, divides the circle into two equal parts. For, in describing the circle, as the distance from the centre *c* is always the same, the distance passed through on one side of the line must be the same as that passed through on the other. Therefore the arc *bae* is equal to the arc *bde*. Now, if *bde* is supposed to revolve on *be*, till it meets the arc *bae*, it will coincide with it; that is, the two arcs will form but one, because they are equal; and the point *d* will coincide with the point *a*, because the angles are right angles. But, if we suppose the arc *abd* to revolve on *ad*, the point *b* will coincide with the point *e*, for the same reason, and the arc *ab* with the arc *ae*. Wherefore the four arcs are equal to one another. Each of these arcs, being the fourth part of a circle, is called *a quadrant.*

The straight line *ad* or *be*, passing through the centre, and terminated both ways by the circumference, is called a diameter of the circle; and the arc *bae* or *bde* is *a semicircle*, or half circle.

302. From the above, it is plain that if two straight lines, meeting in one point, make one and the same straight line, this line produced (continued) will be *a diameter of all the circles which can be described about that point.* Also, that all the angles formed by any number of straight lines meeting in one point, as at *c*, on both sides of a diameter, must be equal *to 4 right angles*, because they will be comprised in the four right angles *bca*, *ace*, *ecd*, and *dcb;* and their number will be equal to *the number of lines meeting in the point c.* If formed on one side of a diameter, they will, of course, be equal *to 2 right angles*, and their number will be *one less than the number of lines meeting in c.*

303. If two points be taken in the circumference of a circle, and we suppose a radius to revolve from one to the other, it is plain that *the radius will pass over the arc and the angle which it subtends* (stretches under) *at one and the same time.* Wherefore, if two arcs be equal, *the angles which they subtend must also be equal:* and, *vice versa*, if the angles be equal, *the arcs must also be equal.*

Now, if 360 lines radiate from the point *c*, on both sides of a diameter, so as to make equal angles with each other, there will be 360 angles. Then, if any number of circles be described about the centre *c*, the circumference of each circle will be divided, by the radiating lines, into 360 equal arcs. These equal angles and equal arcs are called *degrees.* Hence the circumference of every circle, however great or small, is said to consist *of* 360 *degrees*, and a quadrant, or right angle, *of* 90.

304. The arc of a circle intercepted between the extreme points of any two radii, is called *the measure of the angle which they form with each other at the centre.* Thus a quadrant, or 90 degrees, is *the measure of a right angle.* Also it is evident that every arc of a circle is *the same part of the whole circumference of that circle, that the angle which it subtends is of four right angles.*

Geographical Measure.

305. Geography is derived from two Greek words, *Gé*, the earth, and *grapho*, I write; and means *the delineation and description of the earth's surface.*

The exact measurement of the globe we inhabit, is a subject of peculiar interest to the astronomer, geographer, navigator, and engineer, because many important calculations are founded upon it. It has, therefore, been attempted by skilful men of different nations, and though the results have been various, they have differed so little from each other, as to leave no doubt of their near approximation to the truth.

The circumference of the earth in the direction of the meridian, from measures taken in France, is supposed to be *about* 24856 *miles*, and the polar diameter, or axis of the earth, *nearly* 7912. The equatorial circumference *about* 24896 *miles*, and its diameter *nearly* 7925.

Hence a degree on the meridian is 69,04 or $69\frac{2}{45}$ *English or common miles, nearly;* and a degree on the equator *about* $69\frac{1}{6}$.*

306. The degree on the equator, called *a degree of longitude*, from the Latin *longus*, long, (the circumference being longer in that direction than on a meridian,) is divided into 60 equal parts called *minutes*, and, by some, *geographical miles.*

Hence, (288,) a common mile is to a geographical mile as 21600 to 24896, or *very nearly as* 72 *to* 83.

Longitude is either east or west, as it is measured *on the equator*, beginning at the point in which the meridian of some remarkable place cuts that line, which meridian is *in longitude* 0°, and proceeding east or west round the globe to the point where the same meridian cuts the equator again; which last point, being the farthest possible, or opposite extremity of the semicircle, is *in longitude* 180°, *or* 0°, because all places through which the same meridian passes, are said to be *in the same longitude.*

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* Some suppose the mean diameter of the earth to be about 7920 miles, which makes the degree at a mean for the whole surface $69\frac{1}{6}$ miles, nearly.

307. The degree on a meridian is called *a degree of latitude,* from the Latin, *latus,* wide.

Latitude is *north* or *south,* as it is measured *on a meridian towards the North or South pole, beginning at the equator,* which is *in latitude* 0°, and ending at the pole, which is *in lat.* 90°.

Lines passing round the globe, in a direction exactly east and west, and, consequently, parallel to the equator, are called *parallels of latitude;* and all places intersected by any one of these parallels, are said to be *in the same latitude.*

308. Mariners determine the position of their vessel at sea, as well as that of any place on the globe, *by its latitude and longitude.*

Now, though all meridians are equal, the case is very different with parallels of latitude, which diminish as they recede from the equator towards the pole. Hence, if one of these is at a considerable distance from the equator, the degree measured on *it is very different from the equatorial degree.* For example, on the parallel of Philadelphia, a degree is *about* 46 *nautical, or* 53,03 *common miles.*

309. The meridian from which the longitude is reckoned, or *first meridian,* is arbitrary. The French reckon from *that of the Observatory at Paris;* the English and Americans from *that of the Royal Observatory at Greenwich,* near London; and several other nations each from that of its own capital. But this is of little consequence, as it is easy to change one reckoning to the other. For example, the difference of longitude between the French and English meridians is 2° 20′ 15″; and, as the French meridian is east of the English, we say that it is in longitude 2° 20′ 15″ E., while the French consider the English meridian in longitude 2° 20′ 15″ W. Wherefore, to change French longitude to English, *add* 2° 20′ 15″ *to the east longitude, and subtract it from the west.*

310. As the equator, as well as every circle or parallel of latitude, is divided into 360 degrees, and all these are, by the revolution of the earth from west to east, brought successively to the mean solar noon exactly in 24 hours, it is plain

that the noon of each degree in any of those circles, (where day and night alternate in one revolution,) will precede the noon of the next degree to the westward, by the 360th part of 24 hours; that is, *by* 4 *minutes of solar time.*

Hence, if we know the longitude of two places, we have only to *multiply their difference of longitude* in degrees by* 4, *which will give the difference of time by the clock at those places, in minutes.* This time, subtracted from the time at the easternmost place, will show that of the westernmost; or, added to that of the last, will give that of the first. For example, suppose that it is noon at the Observatory at Greenwich, and that we would know the time at that of Paris:

First, $2° 20' 15'' = 2\frac{81}{240}° = \frac{187}{80}°$; then, $\frac{187}{80} \times \frac{4}{1} = \frac{187}{20}$ = 9 m. 21 sec.

Hence, as Paris is E. of Greenwich, it is 9 *m.* 21 *sec. past noon at Paris.* When it is noon at the Paris Observatory, it will, of course, be 23 *h.* 50 *m.* 39 *sec. of the preceding day* at Greenwich; for it is thus that astronomers reckon the hours, *from noon of one day to noon of the next.* In common phrase, we should say 50 m. 39 sec. past eleven A.M., of the same day.

Examples.

1. When it is noon at Philadelphia, in longitude 75° 16′ W. of Greenwich, what is the time at Paris?

Answer, 5 h. 10 m. 25 sec. past noon.

2. Being noon at Philadelphia, in long. 75° 16′ W., what is the difference of time, and what o'clock is it at each of the following places, taking them in an easterly order round the globe, from Philadelphia to Philadelphia again: Figueira, in Portugal, in long. 8° 52′ W.; Lemnos, in the Grecian Archipelago, long. 25° 15′ E.; Pekin, long. 116° 27′ 30″ E., and Cape Mendocino, in long. 124° 7′ W.

		h.	m.	sec.
Answer:	Between Philadelphia and Figueira........	4	25	36
	" Figueira and Lemnos..............	2	16	28
	" Lemnos and Pekin.................	6	4	50
	" Pekin and Cape Mendocino.......	7	57	42
	" Cape Mendocino and Philadelphia	3	15	24

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* When the longitudes are both E. or both W., *their difference is the difference of longitude;* but, when one is E. and the other W., *their sum is the difference of longitude.* When this sum exceeds 180°, *subtract it from* 360°, and *the remainder will be the difference of longitude.*

Also, it is at Figueira past noon	4	25	36
" Lemnos	6	42	4
" Pekin	12	46	54
" Cape Mendocino	20	44	36

Or, in other words, if it is noon on the first of June at Philadelphia, it is, at Figueira, 25 m. 36 sec. past 4 in the afternoon, or 4 h. 25 m. 36 sec., P.M.;* at Lemnos, 6h. 42 m. 4 sec., P.M.; at Pekin, it is 46 m. 54 sec. past midnight, or, June 2nd, 46 m. 54 sec., A.M.; at Cape Mendocino, it is 8 h. 44 m. 36 sec., A.M.

3. Suppose that the above places are all situated on the parallel of 40° 4′ N.,—a degree on this parallel being 45,916 nautical or geographical miles—what are the respective distances of those places in such miles, proceeding eastward, as in the preceding example?

	Nautical miles.
From Philadelphia to Figueira	3048,87 —
" Figueira to Lemnos	1566,52 +
" Lemnos to Pekin	4187,75 —
" Pekin to Cape Mendocino	5483,60 —
" Cape Mendocino to Philadelphia	2243,03 —

4. Reduce the distances in the preceding example to common miles.

	English miles.
Answer: From Philadelphia to Figueira	3514,67
" Figueira to Lemnos	1805,85
" Lemnos to Pekin	4827,55
" Pekin to Cape Mendocino	6321,37
" Cape Mendocino to Philadelphia	2585,72

SECTION XV.

REDUCTION.

311. THIS is the method of finding the equivalent of any number of units of a higher grade in units of a lower, or of any number of units of a lower grade in terms of those of a

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[*] P.M., *Post Meridiem*, Latin, afternoon. A. M., *Ante Meridiem*, forenoon.

higher. The first of these two is called *reduction descending*, and the last, *reduction ascending*.

312. By reduction descending, which is performed *by multiplication*, we find the equivalent of a compound number in units of its lowest order, thus: descending in regular succession, from the highest order to the lowest, we reduce the units of each higher order to those of the next lower, *by multiplying them by the number which shows the ratio of the unit of the higher to that of the lower*, taking care *to add the digits of the lower order* to those of the product, which are of the same kind. For example, to reduce £5 16 s. 4 d. to pence, we proceed thus:

$$\begin{array}{rrr} \pounds & \text{s.} & \text{d.} \\ 5 & 16 & 4 \\ 20 & & \\ \hline 116 & & \\ 12 & & \\ \hline 1396 & & \end{array}$$

In multiplying by 20, it is easy to see that the unit figure of the shillings to be added, will always be the unit figure of the product; we therefore say 0, but 6 is 6; then, twice 5 is 10, and 1 is 11, which gives 116 s. for the value of £5 16 s.

In multiplying by 12, we say, 12 times 6 is 72, and 4 is 76; *six* and go 7: then 12 times 11 is 132, and 7 is 139. Thus we have 1396 for the number of pence, equivalent to £5 16 s. 4 d.

313. By reduction ascending, which is performed *by division*, we find the different denominations of a compound number, from their equivalent expressed in units of its lowest order.

For example, to have the value of 1396 pence in pounds, shillings, and pence: as 12 d. make 1 s., it is evident that as often as 12 is contained in 1396, there will be so many shillings; we therefore divide by 12, and have 116 s. 4 d. for the quotient.

Again, to bring 116 s. to pounds, we divide by 20—because 20 s. make £1—and we have £5 16 s. for the quotient; which, together with the 4 d., gives £5 16 s. 4 d. for the value of 1396 d., as was required.

The operation stands thus:

$$\begin{array}{r|l} 12) & 1396 \\ \hline 2{,}0) & 11{,}6\ \text{s.}\ 4\ \text{d.} \\ \hline & \pounds 5\ 16\ \text{s.}\ 4\ \text{d.} \end{array}$$

To divide by 20, we divide first by 10, and then by 2. Now, to divide by 10, as we separate, by a comma, one figure to the right, the figure separated is *tenths;* but, in dividing these tenths by 2, they will (165) become twentieths—that is, twentieths of a pound, or shillings: wherefore, we consider the figure separated as signifying a number of shillings. In dividing the figures on the left of the comma by 2, if 1 remains, as this, divided by 2, is £$\frac{1}{2}$ = 10 s., we always write this remainder 1, as ten, on the left of the figure separated. In reducing cwts. to tons, we divide by 20 in the same manner.

Hence, we see that, to reduce units of a higher order to those of a lower, *we multiply by the number which shows how many of the lower make one of the higher;* and, to reduce units of a lower order to those of a higher, *we divide by that same number.*

Again, to reduce 4 t. 13 cwt. 3 qrs. 14 lbs. to pounds, the operation is as follows:

$$\begin{array}{rlll}
\text{T.} & \text{cwt.} & \text{qrs.} & \text{lbs.} \\
4 & 13 & 3 & 14 \\
\underline{20} & & & \\
93 & \multicolumn{3}{l}{\text{value in cwts. of 4 t. 13 cwt.}} \\
\underline{4} & & & \\
375 & \multicolumn{3}{l}{\text{value in qrs. of 4 t. 13 cwt. 3 qrs.}} \\
\underline{28} & & & \\
3014 & & & \\
\underline{750} & & & \\
10514 & \multicolumn{3}{l}{\text{value in lbs. of 4 t. 13 cwt. 3 qrs. 14 lbs.}}
\end{array}$$

To reduce 10514 lbs. to tons, we proceed thus:

$$\begin{array}{rl}
28\left\{\begin{array}{r} 7\underline{)\,10514} \\ 4\underline{)\ 1502} \end{array}\right. & \\
4\underline{)\ 375} & -\ 2 \times 7 = 14\ \text{lbs.} \\
2{,}0\underline{)\ 9{,}3} & -\ 3\ \text{qrs.} \\
& 4\ \text{t.}\ 13\ \text{cwt.}\ 3\ \text{qrs.}\ 14\ \text{lbs.}
\end{array}$$

We here first divide by 28, because 28 lbs. make 1 qr.; then by 4, because 4 qrs. make 1 cwt.; and, lastly, by 20, bebecause 20 cwt. makes 1 t.; observing that the 3 which we separate in dividing by 10, being 3 tenths, will, (165,) in dividing by 2, become 3 twentieths; we therefore consider it cwts. : also, the unit which remains in dividing by 2, being $\frac{1}{2}$ t. is 10 cwt., and is therefore placed on the left of 3, making 13 cwt.

Examples.

1. In 7s. 11$\frac{1}{2}$ d., how many farthings? Answer, 382.

2. In 19 s. 11$\frac{3}{4}$ d., how many farthings? Answer, 959.

3. How many barleycorns will reach round the earth on the equator? (305.) Answer, 4732231680.

4. How many seconds are there in a solar year? (297.) Answer, 31556931.

5. How many cubic inches in a cubic mile? Answer, 254358061056000.

6. In 29 t. 17 cwt. 3 qrs. 27 lbs. 15 oz. 13 dr., how many drachms? Answer, 17145853.

7. In 382 farthings, how many shillings? Answer, 7 s. 11$\frac{1}{2}$ d.

8. In 959 farthings, how many shillings? Answer, 19 s. 11$\frac{3}{4}$ d.

9. In 4732231680 barleycorns, how many miles? Answer, 24896.

10. Reduce 31556931 seconds to days. Answer, 365 d. 5 h. 48 m. 51 sec.

11. Reduce 254358061056000 cubic inches to cubic miles. Answer 1.

12. Reduce 17145853 drachms, Avoirs., to tons. Answer, 29 t. 17 cwt. 3 qrs. 27 lbs. 15 oz. 13 dr.

314. If, under a given number of units of any order, we write, in the form of a fraction, the number which shows how many of those units are required to make a unit of some higher order; as this expresses the division of the upper number by the lower, it is plain (313) that *this fraction, whether proper or improper, is a fraction of the higher order;* because, as often as the given number contains this divisor, so often does it contain a unit of the higher order. Thus, to reduce 7 d. to the fraction of a shilling, as 12 d. make 1 s., we say 7 d. $= \frac{7}{12}$ s. To reduce 7 d. to the fraction of a pound: as 240 d. make £1, we say, 7 d. $= \frac{7}{240}$ £. To reduce 440 yds. to the fraction of a mile, as the mile is 1760 yds., we place this number under 440, and have $\frac{440}{1760} = \frac{1}{4}$, for the required fraction. In effect, this is nothing else than to divide 440 yds. by 1760. But, (313,) in dividing yards by 1760, we reduce them to miles; the fraction $\frac{440}{1760}$, or $\frac{1}{4}$, is therefore the fraction of a mile.

315. From what has been said, we can easily reduce all the inferior parts of a compound number to a fraction of the principal unit. For example, to reduce 17 s. 6 d. to the fraction of a pound, we first reduce 6 d. to the fraction of a shilling, which (314) gives $\frac{6}{12} = \frac{1}{2}$ s. We then have $17\frac{1}{2}$ s., or $\frac{35}{2}$ s. Now to reduce $\frac{35}{2}$ s. to the fraction of a pound; we divide by 20—that is, (165,) we multiply the denominator by 20—which gives $\frac{35}{40}$, or $\frac{7}{8}$, for the fraction required.

Or thus: we reduce 17 s. 6 d. to pence, which gives 210 d., under which we place 240, the number of pence in a pound, and have $\frac{210}{240} = \frac{7}{8}$, as before. Also, 210 d. and 240 d. are homogeneous numbers; and, (188,) by placing 210 over 240, we show what part 210 is of 240. But 210 d. is the value of 17 s. 6 d., and 240 d. that of £1: therefore, $\frac{210}{240}$, or $\frac{7}{8}$, is the part or fraction that 17 s. 6 d. is of £1.

Again, to reduce 2 fur. 26 p. 3 yds. 2 ft. to the fraction of a mile, we may proceed thus:

$$2\text{ ft.} = \tfrac{2}{3}\text{ yd.};\ 3\tfrac{2}{3}\text{ yds.} = \tfrac{11}{3}\text{ yd.};\ \tfrac{11}{3} \div 5\tfrac{1}{2} = \tfrac{11}{3} \div \tfrac{11}{2}$$
$$= \frac{\cancel{11}}{3} \times \frac{2}{\cancel{11}} = \tfrac{2}{3}\text{ p.}$$

$$26\tfrac{2}{3}\text{ p.} = \tfrac{80}{3}\text{ p.};\ \tfrac{80}{3} \div 40 = \tfrac{2}{3}\text{ fur.};\ 2\tfrac{2}{3}\text{ fur.} = \tfrac{8}{3}\text{ fur.};$$
$$\text{lastly, } \tfrac{8}{3} \div 8 = \tfrac{1}{3}\text{ m.}$$

Wherefore, the compound number 2 fur. 26 p. 3 yds. 2 ft. is $\frac{1}{3}$ of a mile; or thus:

$$2\text{ fur. }26\text{ p. }3\text{ yds. }2\text{ ft.} = 1760\text{ feet, and }1\text{ m.} = 5280\text{ ft.}:$$
$$\text{then } \tfrac{1760}{5280} = \tfrac{1}{3}\text{ of a mile, as before.}$$

Hence we see that, to reduce the lower denominations of a compound number to a fraction of the principal unit, *we reduce its lowest denomination to a fraction of the next higher, to which we annex it. We then first reduce this mixed number to an improper fraction, and then to a proper fraction of the next higher order, to which we annex it, and thus we proceed till we arrive at the fraction required.*

Or, *we reduce the whole to units of its lowest order, and place the result over the number of units of this order contained in the principal unit, reducing the fraction, if necessary, to its lowest terms.*

The scholar may operate by both methods, which will serve as proof.

Examples.

1. Reduce 19 s. $5\frac{1}{4}$ d. to the fraction of a pound. Answer, $\frac{311}{320}$.

2. Reduce 19 s. $10\frac{3}{4}$ d. to the fraction of a pound. Answer, $\frac{191}{192}$.

3. Reduce 12 s. $9\frac{3}{4}$ d. to the fraction of a pound. Answer, $\frac{41}{64}$.

4. What part of a cwt. is $\frac{7}{8}$ of an ounce Avoirdupois? Answer, $\frac{1}{2048}$.

5. What part of a ton is 5 cwt. 2 qrs. 24 lb.? Answer, $\frac{2}{7}$.

6. What part of a league is 2 m. 7 fur. 39 p. 4 yds. 2 ft. $2\frac{1}{2}$ in.? Answer $\frac{6911}{6912}$.

7. Reduce 3 R. 6 P. 15 s. yds. $5\frac{1}{4}$ s. ft. to the fraction of an acre. Answer, $\frac{835}{1056}$.

8. What is the ratio of 1 R. 23 P. 28 s. yds. $3\frac{3}{4}$ s. f. to 3 R. 39 P. 25 s. yds. 6 s. f.? Answer, $\frac{2}{5}$.

316. Having seen (158) that a fraction may be considered as expressing the division of its numerator, as an integral number, by its denominator, we can, by performing this division, find the value of a fraction of the principal unit, or of a unit of any superior denomination, in terms of the inferior denominations. For example, to find the value of $\frac{791}{960}$ of a pound, as this is the same as the nine hundred and sixtieth part of £791, we divide £791 by 960 in the following manner:

```
       791
        20
     -----
960) 15820 (16 s.  5¾ d.
       960
     -----
      6220
      5760
     -----
       460
        12
     -----
960)  5520 (5 d.
      4800
     -----
       720
         4
     -----
960)  2880 (3 qrs. or ¾ d.
      2880
     -----
      ****
```

It is evident that, when the given fraction is proper, we must reduce the numerator to the next lower denomination before it can contain the denominator; and, if the product is yet too small, to the next lower, &c. In the above example, therefore, we first reduce £791 to shillings, and dividing by 960, we have 16 s. for the quotient, with a remainder of 460 s.: this remainder, reduced to pence, is 5520 d.; and, dividing by 960, we have 5d. for the quotient, and a remainder of 720 d.: this remainder, reduced to farthings, is 2880 qrs.; and, dividing by 960, we have 3 qrs., or $\frac{3}{4}$ d. without remainder. The value of £$\frac{791}{960}$ is, therefore, 16 s. 5$\frac{3}{4}$ d.

This operation may also serve as an illustration of the method of dividing any compound number by a whole number.

Or thus: $\frac{960}{20} = 48$) 791 (16 s. 5$\frac{3}{4}$ d.

$$\begin{array}{r} 48 \\ \hline 311 \\ 288 \\ \hline \frac{48}{12} = 4)\ 23 \\ \hline 5\frac{3}{4}\text{ d.} \end{array}$$

Here, as the divisor 960 is divisible by 20, we divide, and have 48, which we take for a divisor. Now 15820 = 791 × 20, and 960 = 48 × 20. But, (165,) in dividing the dividend and divisor both by the same number, the quotient is not altered: wherefore, 791 ÷ 48 will give the same quotient as 15820 ÷ 960. In like manner 23 ÷ 4 will give the same quotient as 5520 ÷ 960, because 5520 = 23 × 240 and 960 = 4 × 240.

Therefore, in all divisions like the above, *whenever the multiplier of the remainder will divide the divisor, we may so divide*, which will very much shorten the operation.

In the above example, the scholar may observe that, as 960 qrs. make £1, the numerator 791 represents a number of farthings, which may be reduced thus:

$$\begin{array}{r} 4)\ 791 \\ \hline 12)\ 197\frac{3}{4} \\ \hline 16\text{ s. }5\frac{3}{4}\text{ d.} \end{array}$$

Examples.

1. What is the value of £$\frac{311}{320}$? Answer, 19 s. 5$\frac{1}{4}$ d.
2. What is the value of £$\frac{41}{64}$? Answer, 12 s. 9$\frac{3}{4}$ d.

3. What is the value of £$\frac{191}{192}$? Answer, 19 s. 10$\frac{3}{4}$ d.

4. What is the value of $\frac{1}{2048}$ of a cwt.? Answer, $\frac{7}{8}$ oz.

5. What is the value of $\frac{2}{7}$ of a ton?
Answer, 5 cwt. 2 qrs. 24 lbs.

6. What is the value of $\frac{6911}{6912}$ of a league?
Answer, 2 m. 7 fur. 39 p. 4 yds. 2 ft. 2$\frac{1}{2}$ in.

7. What is the value of $\frac{835}{1056}$ of an acre?
Answer, 3 R. 6 P. 15 s. yds. 5$\frac{1}{4}$ s. f.

8. What is the value of $\frac{5}{7}$ of a mile?
Answer, 5 fur. 28 p. 3 yds. 0 ft. 5$\frac{1}{7}$ in.

317. We can, with equal facility, reduce the lower denominations of a compound number to a decimal fraction of the principal unit, and this again to the terms of the lower denominations.

This is done in the same manner as in the preceding articles, except that we divide or multiply decimally. For example, to reduce 9 d. to the decimal fraction of a shilling, we first reduce 9 d. to the vulgar fraction of a shilling, which gives $\frac{9}{12}$ or $\frac{3}{4}$ s.; then, reducing $\frac{3}{4}$ to a decimal, we have ,75 for the fraction required.

Or, we divide 9 immediately by 12, thus:

12) 9,00
,75

In the same manner units of any inferior denomination are reduced to the decimal of a higher denomination.

To find the value, in pence, of ,75 of a shilling, we multiply by 12, thus:

,75
12
9,00

and we have 9 d. Each operation, therefore, proves the other.

To reduce 5 s. 6$\frac{1}{2}$ d. to the decimal of a pound, we operate thus:

2) 1,0
12) 6,5000 d.
20) 5,541$\dot{6}$ s.
£ ,27708$\dot{3}$

Having found ,5 the value of $\frac{1}{2}$ d., we prefix the 6d., and have 6,5 d. for the value of $6\frac{1}{2}$ d.; then, to have the value of 6,5 d. in the decimal of a shilling, we divide by 12, and have $,541\dot{6}$ s.; to which prefixing the 5 s. we have $5{,}541\dot{6}$ s. for the value of 5 s. $6\frac{1}{2}$ d. Lastly, to reduce $5{,}541\dot{6}$ s. to the decimal of a pound, we divide by 20, as in Art. 313; that is, we remove the comma, or suppose it to be removed, one place towards the left; and, dividing $,5541\dot{6}$ by 2, we have $,27708\dot{3}$ of a pound for the value of 5 s. $6\frac{1}{2}$ d., as was required.

Again, we find the value of £ $,27708\dot{3}$ in terms of the lower denominations, thus:

$$\begin{array}{r}
2{,}7708\dot{3}\\
2\\ \hline
5{,}5416\dot{6}\\
12\\ \hline
6{,}50000\\
4\\ \hline
2{,}0
\end{array}$$

To find the value of £ $,27708\dot{3}$ in shillings, we remove the comma one place towards the right, and multiply by 2, which is the same as to multiply by 10, and again by 2—that is to say, by 20; after which, as there are 5 decimals in the number $2{,}7708\dot{3}$, we separate as many in the product, and have $5{,}5416\dot{6}$ s., which is the same as 5 s. and $,5416\dot{6}$ s. Then, to find the value of $,5416\dot{6}$ s. in pence, we multiply by 12, and have 6,5 d. Lastly, we multiply ,5 d. by 4, and have 2 qrs., or $\frac{1}{2}$ d. for the product. Wherefore, collecting the different denominations we have thus found, we have 5 s. $6\frac{1}{2}$ d. for the required value. This last operation proves the preceding one.

In a similar manner, we reduce the inferior terms of a compound number of any other species to the decimal of a higher denomination, and this decimal, again, to the inferior terms. Also, we may, in any case, first *reduce the given compound number to a vulgar fraction of the principal unit, and then reduce this fraction to a decimal fraction.*

Examples.

1. Reduce 19 s. $5\frac{1}{4}$ d. to the decimal of a pound.
Answer, £ ,971875.

2. Reduce 19 s. $10\frac{3}{4}$ d. to the decimal of a pound.
Answer, £ $,994791\dot{6}$.

3. Reduce 12 s. $9\frac{3}{4}$ d. to the decimal of a pound.
Answer, £ ,640625.

4. Reduce $\frac{7}{8}$ oz. Avoirdupois to the decimal of a cwt.
Answer, ,00048828125.

5. Reduce $\frac{5}{7}$ of an ell English to the decimal of a yard.
Answer, $,89\dot{2}8571\dot{4}$.

6. Reduce 2 qrs. $3\frac{3}{7}$ n. to the decimal of an ell English.
Answer, $,\dot{5}7142\dot{8}$.

7. Reduce 28 s. yds $2\frac{4}{7}$ s. f. to the decimal of a square pole.
Answer, $,\dot{9}3506\dot{4}$.

8. Find the value of $,669791\dot{6}$ of a pound sterling.
Answer, 13 s. $4\frac{3}{4}$ d.

9. Find the value of $,\dot{8}5714\dot{2}$ of an ell English.
Answer, 1 yd. 0 qrs. $1\frac{1}{7}$ n.

10. Find the value of $,06\dot{4}2857\dot{1}$ of an acre.
Answer, 10 P. 8 s. yds. $5\frac{11}{14}$ s. f.

11. Find the value of $,\dot{4}2857\dot{1}$ of a league.
Answer, 1 m. 2 fur. 11 p. 2 yds. 1 ft. $0\frac{6}{7}$ in.

12. Required the value of $,\dot{2}8571\dot{4}$ of a ton?
Answer, 5 cwt. 2 qrs. 24 lbs.

318. We have seen (256) that the decimal value of a fraction, the denominator of which is prime to 10, will be *infinite;* that is, (251,) it will be *a repeating decimal.* The decimal value of a proper fraction, therefore, having 7 for its denominator, will be infinite. Now, in dividing a series of nines by 7, we find (257) that the period will contain 6 places of decimals. Hence, in dividing a unit by 7, there must (251) be a different remainder at each division, till we reach the limit of the period; consequently the several remainders must be 1, 2, 3, 4, 5, 6. Wherefore, in dividing by 7 any of the numbers 1, 2, 3, 4, 5, 6, the same figures will always succeed each other, as soon as we find, for remainder, the figure first divided:

consequently, in dividing by 7, any number to which it is prime, the decimal period, which (260) begins at the comma, will (251) always consist of the same 6 figures; and these will succeed each other in the same order, although the period will not always commence with the same figure. Thus we find that

$\frac{1}{7} = ,\dot{1}4285\dot{7}$

$\frac{2}{7} = ,\dot{2}8571\dot{4}$

$\frac{3}{7} = ,\dot{4}2857\dot{1}$

$\frac{4}{7} = ,\dot{5}7142\dot{8}$

$\frac{5}{7} = ,\dot{7}1428\dot{5}$

$\frac{6}{7} = ,\dot{8}5714\dot{2}$

$\frac{8}{7} = 1,\dot{1}4285\dot{7}$

$\frac{9}{7} = 1,\dot{2}8571\dot{4}$

Hence, the student will perceive that the twelfth example (317) may be performed as follows:

Because $,\dot{2}8571\dot{4}$ t. $= \frac{2}{7}$ of a ton, or (158) the seventh part of 2 tons, we divide 2 tons by 7, thus:

$$\begin{array}{r|rrrr} & \text{T.} & \text{cwt.} & \text{qrs.} & \text{lbs.} \\ 7) & 2 & 0 & 0 & 0 \\ \hline & 0 & 5 & 2 & 24 \end{array}$$

Wherefore $,\dot{2}8571\dot{4}$ t. = 5 cwt. 2 qrs. 24 lbs.

By operating decimally, thus:

$$\begin{array}{rl} ,\dot{2}8571\dot{4} & \\ 20 & \\ \hline 5,\dot{7}1428\dot{5} & \text{(See Art. 267)} \\ 4 & \\ \hline 2,\dot{8}5714\dot{2} & \text{(267)} \\ 28 & \\ \hline 6,\dot{8}5714\dot{2} & \\ 17,142857 & \text{(See Art. 268)} \\ \hline 23,\dot{9}9999\dot{9} & \end{array}$$

we find the same result. For, as $,\dot{9}9999\dot{9}$ is the same as its denominator, it is equal to a unit; therefore $23,\dot{9}9999\dot{9} = 24$.

SECTION XVI.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF COMPOUND NUMBERS.

Addition of Compound Numbers.

319. We write the given numbers under each other, so that units of the same kind may be in the same column, and commence by adding the units of the lowest denomination. If their sum does not contain a sufficient number of units to compose a unit of the next higher denomination, we write it under its kind: but if it contains as many units as will make one or more, we reduce it (313) to that denomination, writing the remainder, if any, under the units of its kind, and adding the figures of the quotient to those of the same order in the next higher denomination, with which we proceed in like manner.

Example 1.

£	s.	d.
25	16	6
332	19	11
64	11	9
58	9	6
481	17	8

As the sum of the pence is 32, which (313) gives 2s. 8d., we write 8 under pence, and add 2 to the unit column of shillings: then as the sum of this is 27, we write 7, and carry 2 to the column of tens. The sum of this is 5, and as each unit in this 5 is 10 s., two of these will make £1. We therefore take the half of 5 for pounds, which gives 2 and 1 over. This 1, which is 10 s., we place on the left of 7, and carry the £2 to the column of pounds.

Example 2.

A.	R.	P.	s. yds.	s. f.
3	3	36	26	8
36	2	27	19	7
46	3	13	25	5
16	3	35	21	6
25	1	12	12	4
129	3	6	15	$5\frac{1}{4}$

Here, the sum of the s. f. is 30, which, reduced (313) to s. yds., gives 3 s. yds. 3 s. f. We therefore write 3 s. f. under the column of feet, and carry 3 s. yds. to the column of yards. The sum of the yards is 106; and, as $30\frac{1}{4}$ make a pole, we divide 106 by $30\frac{1}{4} = \frac{121}{4}$; that is, we multiply 106 by $\frac{4}{121}$, which gives $\frac{424}{121}$, or $3\frac{61}{121}$ P. Reducing this fraction of a pole to s. yds., we have $\frac{61}{121} \times \frac{121}{4} = \frac{61}{4} = 15\frac{1}{4}$ s. yds. We have, then, for the sum of the yards, 3 P. $15\frac{1}{4}$ s. yds. We, therefore, write 15 under s. yds.; but, as it would be awkward to leave a fraction in connection with the s. yds. while we have a lower denomination in the question, we reduce $\frac{1}{4}$ s. yd. to feet, in multiplying by 9, which gives $\frac{9}{4} = 2\frac{1}{4}$ s. f.; and adding this to the 3 s. f. first written, we have $5\frac{1}{4}$ s. f., which we leave under that column. Then, carrying 3 to the column of poles, we find the sum 126, and, reducing this to roods, we have 3 R. 6 P. Writing 6 under poles, and carrying 3 to the roods, we find the sum 15, which, reduced to acres, gives 3 A. 3 R. Lastly, we write 3 under roods, and carry 3 to the column of acres, the sum of which is 129. Having completed the addition, the sum total is 129 A. 3 R. 6 P. 15 s. yds. $5\frac{1}{4}$ s. f.

3. Required the sum of the four last numbers of the preceding example. Ans. 125 A. 3 R. 9 P. 18 s. yds. $8\frac{1}{2}$ s. f.

By adding this sum to the first number of Example 2, we shall again have 129 A. 3 R. 6 P. 15 s. yds. $5\frac{1}{4}$ s. f., and this serves as proof of the correctness of the first addition.

In the same manner the scholar may prove the following

Examples.

£	s.	d.
11	11	$9\frac{3}{4}$
78	16	$2\frac{1}{4}$
89	12	$11\frac{1}{2}$
65	19	$7\frac{3}{4}$
14	13	$6\frac{1}{4}$
33	15	$9\frac{3}{4}$

£	s.	d.
79	12	$3\frac{1}{2}$
32	14	$7\frac{3}{4}$
59	17	$8\frac{1}{4}$
43	10	$2\frac{3}{4}$
86	4	$11\frac{1}{2}$
97	13	$6\frac{3}{4}$

£	s.	d.
23	19	$4\frac{1}{2}$
31	6	$11\frac{1}{4}$
49	18	$7\frac{3}{4}$
63	13	$10\frac{1}{4}$
77	12	$9\frac{1}{2}$
84	17	$5\frac{3}{4}$

cwt.	qrs.	lbs.	oz.	drs.
13	2	18	12	8
29	1	23	14	9
34	3	27	15	11
67	3	19	12	15
45	1	25	14	7
23	3	26	11	6

A.	R.	P.	s. yds.	s. f.
7	2	31	29	8
9	3	28	19	7
8	2	17	15	6
9	3	23	25	8
6	2	39	24	7
4	3	32	16	5

bar.	gal.	qt.
29	27	3
13	15	3
44	11	2
52	26	3
89	29	2
93	14	1

Subtraction of Compound Numbers.

320. We place the less number under the greater in the same manner as for addition; and, commencing with the lowest order, we subtract the units of each order in the less number from the corresponding units of the greater, writing each remainder under the order which gave it. But, if the number of units of any inferior order of the less number is greater than the number above it, we *add as many units to the upper number as will make a unit of the next higher order*, and, having subtracted the lower number from the sum, we *add a unit to the next higher order of the lower number.* The numbers are thus both increased by the same quantity, which (84) does not affect their difference.

Examples.

	A.	R.	P.	s. yds.	s. f.
From	129	3	6	15	5$\frac{1}{4}$
take	125	3	9	18	8$\frac{1}{2}$
	3	3	36	26	8

Here, as we cannot subtract $\frac{1}{2} = \frac{2}{4}$ from $\frac{1}{4}$, we borrow a unit from the 5 s. f., which is equal to $\frac{4}{4}$: then $\frac{4}{4} + \frac{1}{4} = \frac{5}{4}$, and $\frac{5}{4} - \frac{2}{4} = \frac{3}{4}$, which we write underneath. Having borrowed a unit from the 5 s. f., instead of diminishing this by a unit, we add a unit (84) to the 8, which makes 9: then, as 9 is greater than 5, we borrow a s. yd., reducing this to s. f., and adding it to 5, we have 14: then $14 - 9 = 5$, which we write under s. f. Having borrowed a s. yd., we add 1 to 18, which makes 19: then as 19 is greater than 15, we borrow a pole, which, reduced to yards, and added to 15, makes 45$\frac{1}{4}$: subtracting 19, there remains 26$\frac{1}{4}$: wherefore, we write 26 underneath, and reducing $\frac{1}{4}$ s. yd. to s. f. we have 2$\frac{1}{4}$, which added to the 5$\frac{3}{4}$ s. f. already found, gives 8. This we leave under the column of s. f. Having borrowed a pole, we add 1 to 9, which makes 10: we then say, not 10 from 6, but borrowing a rood or 40 poles, 10 from 46, leaves 36, which we write underneath. We then add 1 to 3, which makes 4, and borrowing an acre, or 4 R., we say, 4 from 7 leaves 3, which we write underneath. Lastly, having borrowed an acre, we add 1 to 5, which makes 6, and 6 from 9 leaves 3, which, when written underneath, completes the operation.

This example is a proof of the second example, (319,) the greater number being the sum total of the five numbers added, the less, the sum of the four last numbers, and the remainder, the first number.

Note. We may, in all cases, *reduce both numbers to a vulgar or decimal fraction of the principal unit, and find, after subtraction, the value of the result.*

The student may prove the following examples by addition and subtraction:

2. From $\frac{1}{6}$ of $\frac{3}{4}$ of £1, take $\frac{5}{7}$ of $\frac{7}{8}$ of 1 s.
Answer, 1 s. 10$\frac{1}{2}$ d.

3. From 3 t. 0 cwt. 2 qrs. 5 lbs. 13 oz., take $\frac{2}{3}$ of $\frac{5}{6}$ of 1 cwt.
Answer, 2 t. 19 cwt. 3 qrs. 27 lbs. 9 oz. 7$\frac{1}{9}$ dr.

4. From 3 A. 0 R. 4 P. 9 s. yds. 3 s. f., take $\frac{1}{3}$ of 1 A.
Answer, 2 A. 2 R. 30 P. 29 s. yds. 4 s. f. 72 s. in.

5. From ,$\dot{4}2857\dot{1}$ of a league, take $\frac{3}{4}$ of $\frac{6}{7}$ of a mile.
Answer, 5 fur. 5 p. 3 yds. 2 ft. 9$\frac{3}{7}$ in.

6. From $\frac{5}{7}$ of a square mile, take 100$\frac{2}{3}$ acres.
Answer, 356 A. 1 R. 36 P. 5 s. yds. 6$\frac{6}{7}$ s. f.

	A.	R.	P.	s. yds.	s. f.
(7.)	23	2	21	19	6
	15	3	35	24	7

	£	s.	d.
(8.)	39	6	8$\frac{1}{2}$
	13	17	9$\frac{3}{4}$

	cwt.	qrs.	lbs.	oz.	drs.
(9.)	22	1	13	7	8
	11	3	17	8	11

	bar.	gal.	qt.
(10.)	13	19	2
	7	29	3

Multiplication of Compound Numbers.

To multiply a compound number by a simple or homogeneous one:

321. A familiar instance, requiring this multiplication is—*when a number of units of the same kind is given, and the* *price of one unit, to find the value of the whole, the price being compound.* It is plain that, in this case, *we must multiply the price of a unit by the number of units.* There are several methods of effecting this: that, however, which seems to be most naturally suggested is, to multiply by the given multiplier, each order of units, from the lowest to the highest; and, when any product exceeds in value a unit of the next higher order, to reduce it to that order, as in addition.

For example, we find the value of 17 yds. at 6 s. 11¾ d. per yd., thus:

$$\begin{array}{r} 6\quad 11\tfrac{3}{4} \\ 17 \\ \hline \pounds 5\quad 18\quad 7\tfrac{3}{4} \end{array}\ \text{value of 17 yds.}$$

Beginning with the farthings, we say, 17 × 3 = 51 qrs. = 12¾ d., we write ¾, and carry 12: then 17 times 11 is 187, and 12 is 199 d. = 16 s. 7 d., we write 7, and carry 16: lastly, 17 times 6 is 102 and 16 is 118 s. = £5 18 s., which we write, and the work is done.

The work may be proved thus:—6 s. 11¾ d. = 335 qrs. = $\pounds\frac{335}{960}$: then $\pounds\frac{335\times 17}{960} = \frac{5695}{960}$ = £5 18 s. 7¾ d.

Or thus: 11¾ d. = 11,75 d. = $,9791\dot{6}$ s., and $6,9791\dot{6}$ s. = £ $,348958\dot{3}$: then £ $,348958\dot{3} \times 17$ = £$5,932291\dot{6}$ = £5 18 s. 7¾ d.

By both these methods the student may prove the following

Examples.

1. Required the cost of 19 yds. of cloth, at 17 s. 6 d. per yd.
Answer, £16 12 s. 6 d.

2. What is the cost of 13 yds. at 18 s. 9¾ d. per yd.?
Answer, £12 4 s. 6¾ d.

3. What is the weight of 17 hhds. of sugar, each containing 9 cwt. 3 qrs. 14 lbs. 8 oz.?
Answer, 8 t. 7 cwt. 3 qrs. 22 lbs. 8 oz.

4. What is the content of a farm consisting of 11 fields, each containing 39 A. 3 R. 23 P. 29 s. yds. 8 s. f. 111 s. in.?
Answer, 438 A. 3 R. 23 P. 27 s. yds. 1 s. f. 141 s. in.

322. When the multiplier is too large to multiply by at once, and can be resolved into 2 or more convenient factors, *we multiply, successively, by all the factors, taking them in any order which may seem most convenient.*

Examples.

1. How much land is there in a township, consisting of 385 farms, each containing 129 A. 3 R. 6 P. 15 sq. yds. $5\frac{1}{4}$ s. f.?

s. m.	A.	R.	P.	s. yds.	s. f.	
0	129	3	6	15	$5\frac{1}{4}$	
					5	. 7 . 11 = 385
1	8	3	32	17	$3\frac{3}{4}$	
					7	
7	62	2	28	0	$8\frac{1}{4}$	
					11	
78	49	1	28	10	$0\frac{3}{4}$	

In the multiplier 385, it is easy to recognize the 3 factors 5, 7, and 11. We therefore multiply by these in succession, and find that the township contains 78 square miles, 49 acres, 1 rood, 28 poles, 10 square yards, and $\frac{3}{4}$ of a square foot, or 108 square inches.

2. How much land is there in a township, consisting of 495 farms, each containing 568 A. 3 R. 22 P. 6 s. yds. $6\frac{1}{3}$ s. f.?

Answer, 440 s. m.

323. When the multiplier cannot be resolved into convenient factors, *take the nearest composite number inferior to it, by which multiply, as in the preceding article: then multiply the given compound number by the difference between the composite number and the given multiplier. This last product, added to the product given by the composite number, is the required product.*

Examples.

1. What weight of coffee is there in 236 bags, each containing 1 cwt. 1 qr. 4 lbs. 5 oz. 6 dr.?

T.	cwt.	qr.	lbs.	oz.	dr.	
0	1	1	4	5	6	
					3	. 7 . 11 + 5 = 236
0	3	3	13	0	2	— 3 times
					7	
1	7	0	7	0	14	— 21 times
					11	
14	17	2	21	9	10	— 231 times
	6	1	21	10	14	— 5 times
15	4	0	15	4	8	— 236 times.

As the factors of 236 are not convenient, seeing that one of them is the prime 59, we take the composite number 231, which differs by only 5 units, and of which it is easy to see that the factors are 3, 7, and 11. By these we multiply as in the preceding article, and to the product 14 t. 17 cwt. 2 qrs. 21 lbs. 9 oz. 10 dr., which is 231 times the multiplicand, we add 6 cwt. 1 qr. 21 lbs. 10 oz. 14 dr., which is 5 times that number. The sum, therefore, 15 t. 4 cwt. 0 qrs. 15 lbs. 4 oz. 8 dr., is 236 times the multiplicand, or whole weight required.

2. How far will a man travel in 534 days, at the average rate of 25 m. 5 fur. 39 p. 4 yds. 2 ft. 6 in. per day?

Answer, 13750 m. 2 fur. 15 p. 1 yd. 1 ft. 6 in.

3. How much grain is there in 785 bags, each containing 2 bush. 3 pks. 1 gal. 3 qts. 1 pt.; and how high will it cover a floor that is 20 ft. long and 15 ft. wide, allowing 2150 cubic inches to the bushel?

Answer. The quantity of grain is 2342 bush. 2 pks. 1 gal. 3 qts. 1 pt.; and the depth to which it will cover the floor is 9 ft. $8\frac{32869}{55296}$ in., or 9 ft. $8\frac{3}{5}$ in., nearly.

324. When a compound number is reduced to its lowest denomination, or to a fraction, proper or improper, of its principal unit, *it becomes simple;* and, in this form, susceptible of all the fundamental operations, *as an abstract number.*

For example, to multiply 13 cwt. 3 qrs. 16 lbs. by 191, we may proceed thus: 13 cwt. 3 qrs. 16 lbs. = 1556 lbs.: then, $1556 \times 191 = 297196$ lbs. = 132 t. 13 cwt. 2 qrs. 4 lbs.

Or thus: 13 cwt. 3 qrs. 16 lbs. = $\frac{389}{560}$ t.: then, $\frac{389 \times 191}{560}$ $= \frac{74299}{560} =$ 132 t. 13 cwt. 2 qrs. 4 lbs., as before

By the usual method:

t.	cwt.	qr.	lbs.
0	13	3	16
			191
132	13	2	4

What weight of iron is there in 171 loads, each weighing 19 cwt. 3 qrs. 12 lbs.? Answer, 169 t. 15 cwt. 2 qrs. 8 lbs.

Multiplication of a Compound Number by a Compound Number:

325. From the above, it is manifest that the multiplication of two compound numbers may be reduced to *the multiplica-*

tion of a fraction by a fraction, thus: Reduce each of the given numbers to a fraction of its principal unit; multiply the two fractions together, and reduce the product to the denominations required by the nature of the question.

Examples.

1. What is the cost of 6 cwt. 3 qrs. 14 lbs. of sugar, at £3 5 s. 6 d. per cwt.?

3 qrs. 14 lbs. $= 3\frac{1}{2}$ qrs. $= \frac{7}{2}$ qrs. $= \frac{7}{8}$ cwt., and $6\frac{7}{8}$ cwt. $= \frac{55}{8}$ cwt. 5 s. 6 d. $= 5\frac{1}{2}$ s. $= \frac{11}{2}$ s. $= \frac{11}{40}$ £, and $3\frac{11}{40}$ £ $= \frac{131}{40}$ £.

Now, as $\frac{55}{8}$ is the given quantity in cwts., and $\frac{131}{40}$ £, the cost of 1 cwt., it is evident that $\frac{131}{40} \times \frac{55}{8} = \frac{131}{8} \times \frac{11}{8} = \frac{1441}{64}$ £ = £22 10 s. $3\frac{3}{4}$ d., is the cost of the whole, as required.

The student may, by the preceding methods, perform the following examples, and prove the work by vulgar and by decimal fractions:

2. 187 yds. at 3 s. 6 d. per yd., amount to £32 14 s. 6 d.
3. 523 yds. " 4 s. 8 d. " " 122 0 8
4. 315 yds. " 5 s. $3\frac{3}{4}$ d. " " 83 13 $5\frac{1}{4}$
5. 735 yds. " 19 s. $1\frac{1}{2}$ d. " " 702 16 $10\frac{1}{2}$
6. 1962 yds. " 23 s. $5\frac{3}{4}$ d. " " 2303 6 $1\frac{1}{2}$
7. 2627 yds. " 22 s. $11\frac{1}{4}$ d. " " 3012 16 $9\frac{3}{4}$

8. 11 cwt. 1 qr. 14 lbs. of sugar, at £3 15 s. 6 d. per cwt., amount to £42 18 s. $9\frac{3}{4}$ d.

9. 4 cwt. 3 qrs. 14 lbs. of sugar, at £2 10 s. 6 d. per cwt., amount to £12 6 s. $2\frac{1}{4}$ d.

10. 7 cwt. 0 qr. 19 lbs. of sugar, at £3 14 s. $8\frac{1}{2}$ d. per cwt., amount to £26 15 s. $7\frac{1}{2}$ d.

11. 13 cwt. 2 qrs. 7 lbs. of sugar, at £2 3 s. $9\frac{3}{4}$ d. per cwt., amount to £29 14 s. $2\frac{1}{2}$ d.

12. 9 cwt. 1 qr. 12 lbs. of sugar, at £5 11 s. $6\frac{3}{4}$ d. per cwt., amount to £52 3 s. $10\frac{3}{4}$ d.

13. 29 cwt. 3 qrs. 17 lbs. of sugar, at £7 1 s. $4\frac{1}{2}$ d. per cwt., amount to £211 7 s. $4\frac{1}{4}$ d.

14. 297 t., at £13 12 s. $0\frac{1}{2}$ d. per t., amount to £4039 16 s. $4\frac{1}{2}$ d.

15. 921 t., at £9 19 s. $1\frac{1}{2}$ d. per t., amount to £9169 14 s. $1\frac{1}{2}$ d.

Division of Compound Numbers.

Division of a Compound Number by a Simple Number:

326. If the dividend and divisor represent quantities of different kinds, we first divide the highest order of units in the dividend by the divisor. We then reduce the remainder of this division to the next lower denomination, adding the units of the dividend which are of this denomination: after which we divide and proceed with the remainder of this second division as with that of the first. We continue thus to the lowest denomination, and, collecting the several quotients, we have the total quotient or compound number sought. If the highest order in the dividend does not contain the divisor, *we reduce it to the next lower order, taking care to add the units of this denomination.*

Examples.

1. If 30 yds. of cloth cost £10 9 s. $4\frac{1}{2}$ d., what is the cost per yd?

£ s. d.
10 9 $4\frac{1}{2}$
20
30) 209 s. $4\frac{1}{2}$ (6 s. $11\frac{3}{4}$ d.
180
29 s.
12
30) 352 (11 d.
330
22 d.
4
30) 90 (3 qrs.
90
**

As 30 is not contained in 10, we reduce the £10 to shillings, and add the 9 s. of the dividend. The sum 209 s. we divide by 30, which gives 6 s. for the quotient and 29 s. for remainder. This remainder we reduce to pence, and add the 4 d. of the dividend. The sum 352 d. we divide by 30, which gives 11 d. for the quotient and 22 d. for remainder. This last we reduce to farthings, and add the 2 farthings of the dividend. The sum 90, we divide by 30, and have 3 farthings for the quotient, without remainder. Lastly, we collect the several quotients, and have 6 s. 11¾ d. for the cost per yd., as was required.

We may also divide by the factors of 30, thus:

$$\begin{array}{rrrr} & \text{£} & \text{s.} & \text{d.} \\ 5) & 10 & 9 & 4\frac{1}{2} \\ \hline 6) & 2 & 1 & 10\frac{1}{2} \\ \hline & \text{£}0 & 6\,\text{s.} & 11\frac{3}{4}\,\text{d.} \end{array}$$

2. If 185 yds. cost £604 6 s. 8 d., what is the cost per yd.? Answer, £3 5 s. 4 d.

3. If 284 yds. cost £700 4 s. 9 d., what is the cost per yd.? Answer, £2 9 s. 3¾ d.

4. If 316 t. cost £4034 5 s. 4 d., what is the cost per ton? Answer, £12 15 s. 4 d.

5. If 297 A. cost £4039 16 s. 4½ d., what is the cost per acre? Answer, £13 12 s. 0½ d.

6. If 921 cwt. cost £9169 14 s. 1½ d., what is the cost per cwt.? Answer, £9 19 s. 1½ d.

7. Divide 78 s. m. 49 A. 1 R. 28 P. 10 s. yds. 0¾ s. f. by 385. Answer, 129 A. 3 R. 6 P. 15 s. yds. 5¼ s. f.

For further practice, the student may reverse the 6 examples which immediately follow the first example of the preceding article; in each of which he may divide the whole cost by the number of yards, which will give *the cost per yard.*

327. When the dividend and divisor represent quantities of the same kind, we must observe whether the quotient should or should not also be of the same kind. If it should, we operate as in the preceding article. For example, suppose that, with £15, we have gained £27 10 s. 3¾ d., and that we would find the gain on each pound. It is plain that this gain should be the fifteenth part of the whole gain, and therefore of the same name with the divisor and dividend: consequently, we divide £27 10 s. 3¾ d. by 15, and have £1 16 s. 8¼ d. for the required gain.

328. But, if the nature of the question requires that the quotient should be of a different kind, we reduce the divisor and dividend to the lowest denomination of the dividend; after which, considering the units of the dividend as being of the same kind with those of the quotient that we seek, we divide as in Art. 326.

For example, if we would know how much land can be bought for £483 5 s. 7½ d., at the rate of £5 per acre, it is evident, from the nature of the question, that the quotient must be acres and parts of an acre, and that the number of these must be determined by the number of times that £5 is contained in £483 5 s. 7½ d. Now, as dissimilar units cannot have a ratio to each other, in order to express the ratio of the dividend to the divisor *both numbers must be rendered homogeneous.* But this can only be done *by reducing both numbers to the lowest denomination of the dividend;* or (324) *the dividend to a fraction of its principal unit.* We therefore reduce the numbers, thus: £483 5 s. 7½ d. = 463950 qrs.; £5 = 4800 qrs. The abstract ratio of these numbers expresses the required number of acres; that is, $\frac{463950}{4800}$ A. = 96 A. 2 R. 25 P., is the number of acres and parts of an acre required by the question. Or thus: 5 s. 7½ d. = £$\frac{9}{32}$: then $483\frac{9}{32}$ = £$\frac{15465}{32}$, and $\frac{15465}{32} \div 5 = \frac{3093}{32}$ A. = 96 A. 2 R. 25 P., as before.

Examples.

1. How much corn, at 8 s. per bushel, can be bought for £26 10 s.? Answer, 66 bush. 1 pk.

2. How much cloth, at 4 s. per yd., can we buy for £20 16 s. 4 d.? Answer, 104 yds. 0 qr. 1⅓ n.

3. At £4 per acre, how much land can be bought for £314 19 s. 6 d.? Answer, 78 A. 2 R. 39 P.

4. Having planted 25 bushels of potatoes, and harvested 437 bush. 2 pks., what is the rate of increase?
Answer, 17 bush. 2 pks. for each bushel, or 17½ for 1.

Division of a Compound Number by a Compound Number:

329. When the divisor and dividend are both compound numbers, but of different kinds, we *reduce the divisor to a fraction of the principal unit, and* divide the dividend by this fraction in its lowest terms; that is, (212,) we *multiply the dividend by the denominator of this fraction, and divide the product by the numerator.*

Examples.

1. If 15 cwt. 2 qrs. 8 lbs. of sugar cost £39 1 s. 2 d., what is it per cwt.?

15 cwt. 2 qrs. 8 lbs. = $15\frac{4}{7}$ cwt. = $\frac{109}{7}$ cwt.

```
         £   s.  d.
         39  1   2
                 7
      ─────────────
109)  273    8   2 (£2 10 s. 2 d.
      218
      ───
       55
       20
     ─────
109) 1108 (10 s.
     1090
     ────
       18
       12
     ────
109)  218 (2 d.
      218
      ───
      ***
```

Having divided the whole cost, £39 1 s. 2 d. by $\frac{109}{7}$, which is the value of the given quantity in cwts., we have, for the value of 1 cwt., £2 10 s. 2 d.

2. If 13 yds. 2 qrs. of cloth cost £2 5 s. 6$\frac{3}{4}$ d., what is it per yd? Answer, 3 s. 4$\frac{1}{2}$ d.

3. If 14 cwt. 3 qrs. 9 lbs. of sugar cost £40 9 s. 6 d., what is it per cwt.? Answer, £2 14 s. 7 d. +

4. If 15 t. 16 cwt. of hay cost £71 2 s., what is it per ton? Answer, £4 10 s.

For more examples the student is referred to examples 8, 9, 10, 11, 12, 13, in Art. 325; in each of which the whole cost, divided by the fractional equivalent of the quantity in cwts., will give the price per cwt.

330. When the divisor and dividend are of the same kind, and the nature of the question requires that the quotient shall, in kind, agree with them, we operate as in the preceding article. But, if the question requires that the quotient shall be of a different kind, we *reduce the dividend and divisor to the lowest denomination mentioned in either*, and divide as before.

Or, we *reduce each of the given compound numbers to a fraction of the principal unit, divide as in fractions, and reduce the quotient as usual.*

Examples.

1. How much sugar can be bought for £39 1 s. 2 d., at £2 10 s. 2 d. per cwt?

Here the divisor and dividend are of the same kind, but the question demands a quotient of a different kind; we therefore say, £39 1 s. 2 d. = 9374 d.

Also £ 2 10 s. 2 d. = 602 d.

Then $\frac{9374}{602}$ is the abstract ratio of the price of the whole quantity sought to the price of a unit of that quantity: but the quantities evidently have the same relation to each other that their prices have: wherefore, as the denominator 602 represents one unit of the quantity sought, or 1 cwt., and the numerator 9374, the quantity itself in units of the same order as those of the denominator, it follows that

$\frac{9374}{602}$ cwt. = 15 cwt. 2 qrs. 8 lbs. is the quantity sought.

Or thus: £39 1 s. 2 d. = $39\frac{7}{120} = \frac{4687}{120}$; and

£ 2 10 s. 2 d. = $2\frac{122}{240} = 2\frac{61}{120} = \frac{301}{120}$:

then, $\frac{4687}{120} \times \frac{120}{301} = \frac{4687}{301}$ = 15 cwt. 2 qrs. 8 lbs., as before.

2. At £3 5 s. 4 d. per yd., how many yards can you buy for £604 6 s. 8 d.? Answer, 185.

3. At £2 9 s. $3\frac{3}{4}$ d. a yard, how many yards can be bought for £700 4 s. 9 d.? Answer, 284.

4. At £12 15 s. 4 d. a ton, how many tons can we buy for £4034 5 s. 4 d.? Answer, 316.

5. At 13 12 s. $0\frac{1}{2}$ d. a ton, how many tons can be bought for £4039 16 s. $4\frac{1}{2}$ d.? Answer, 297.

6. At £9 19 s. $1\frac{1}{2}$ d. a ton, how many tons can be bought for £9169 14 s. $1\frac{1}{2}$? Answer, 921.

7. At £4 10 s. a ton, how much hay can be bought for £71 2 s.? Answer, 15 t. 16 cwt.

8. At £3 15 s. 6 d. per cwt., how much sugar can be bought for £42 18 s. $9\frac{3}{4}$ d.? Answer, 11 cwt. 1 qr. 14 lbs.

9. At £2 10 s. 6 d. per cwt., how much can be bought for £12 6 s. $2\frac{1}{4}$ d.? Answer, 4 cwt. 3 qrs. 14 lbs.

10. At £3 14 s. $8\frac{1}{2}$ d. per cwt., how much can be bought for £26 15 s. $7\frac{1}{2}$ d.? Answer, 7 cwt. 0 qr. 19 lbs.

11. At £2 3 s. $9\frac{3}{4}$ d. per cwt., how much can be bought for £29 14 s. $2\frac{1}{2}$ d.? Answer, 13 cwt. 2 qrs. 7 lbs.

12. At £5 11 s. $6\frac{3}{4}$ d., per cwt., how much can be bought for £52 3 s. $10\frac{3}{4}$ d.? Answer, 9 cwt. 1 qr. 12 lbs.

SECTION XVII.

PRACTICE AND CALCULATIONS BY COMPLEMENTS.

Practice.

331. PRACTICE, so called from its continual application in the purchase and sale of goods, is *an easy and expeditious method of multiplying compound numbers.* It consists in resolving the lower denominations of a compound number into convenient aliquot parts or measures of the principal unit, and of each other; and then ascertaining the value of each aliquot or part, *by taking a corresponding part of the value of the unit or of the value of that part which the aliquot measures.*

332. When one of the given numbers is simple, *either of them may be taken for multiplicand or multiplier.* For example, if we would find the cost of 27 yds. of cloth, at 3 s. 9¾ d. per yd., it is evident that this cost is 27 times 3 s. 9¾ d. But 27 units of whatever kind, at 3 s. 9¾ d. per unit, will give the same result. The number 27, therefore, or multiplier, may be referred to the kind of units that we seek, and thus become the multiplicand; that is, it may be considered as £27, 27 s., or 27 units of any kind to which we can reduce 3 s. 9¾ d. For it is evident (59) that both the given numbers must be referred to the same kind of unit, before each can thus take the place of the other. Assuming £27 for the multiplicand, we take parts of £1 for the multiplier 3 s. 9¾ d., thus:

£	£				s.	d.	
2 s. is $\frac{1}{10}$ of £1	27						
1 s. is $\frac{1}{2}$ of 2 s.	2	14 s.	0 d.	cost of 27 units, at	2	0	per unit.
6 d. is $\frac{1}{2}$ of 1 s.	1	7	0	.... at	1	0	"
3 d. is $\frac{1}{2}$ of 6 d.	0	13	6	.... at	0	6	"
3 qrs. is $\frac{1}{4}$ of 3 d.	0	6	9	.... at	0	3	"
	0	1	$8\frac{1}{4}$	..............and at	0	$0\frac{3}{4}$	"
	£5	2s.	$11\frac{1}{4}$d.	total, or cost at	3	$9\frac{3}{4}$	

333. When any of the lower denominations is too small a part of the preceding aliquot to give a convenient divisor, *we introduce a convenient aliquot, at pleasure, and bar the product* which it gives, *as not belonging to the question:* after which we proceed as usual.

To illustrate this, we may proceed with the above example, thus:

	£	s.	d.	
1 s. 3 d. is $\frac{1}{16}$ of £1	27			
	1	13	9	
	1	13	9	cost, at .... 3 s. 9 d.
3 d. is $\frac{1}{5}$ of 1 s. 3 d.	1	13	9	
3 qrs. is $\frac{1}{4}$ of 3 d....	~~0~~	~~6~~	~~9~~	cost, at 3 d. (barred.)
	0	1	8$\frac{1}{4}$	cost, at............ 0 s. 0$\frac{3}{4}$ d.
	£5	2	11$\frac{1}{4}$	cost, at............ 3 s. 9$\frac{3}{4}$ d.

Or thus:

	£	s.	d.	
2 s. 0 d. is $\frac{1}{10}$ of £1	27			
1 s. 8 d. is $\frac{1}{12}$ of £1	2	14	0	cost, at .... 2 s. 0 d.
1 s. is $\frac{1}{2}$ of 2 s.......	2	5	0	cost, at .... 1 s. 8 d.
1$\frac{1}{2}$ d. is $\frac{1}{8}$ of 1 s.....	~~1~~	~~7~~	~~0~~	cost, at 1 s. (barred)
$\frac{1}{4}$ d. is $\frac{1}{6}$ of 1$\frac{1}{2}$ d.....	0	3	4$\frac{1}{2}$	cost, at .... 0 s. 1$\frac{1}{2}$ d.
	0	0	6$\frac{3}{4}$	cost, at .... 0 s. 0$\frac{1}{4}$ d.
	£5	2	11$\frac{1}{4}$	cost, at .... 3 s. 9$\frac{3}{4}$ d.

Or thus:

	£	s.	d.	
2 s. 6 d. is $\frac{1}{8}$ of £1......	27			
1 s. 3 d. is $\frac{1}{2}$ of 2 s. 6 d.	3	7	6	cost, at......... 2 s. 6 d.
3 d. is $\frac{1}{5}$ of 1 s. 3 d.....	1	13	9	cost, at......... 1 s. 3 d.
3 qrs. is $\frac{1}{4}$ of 3 d........	~~0~~	~~6~~	~~9~~	
	0	1	8$\frac{1}{4}$	cost, at......... 0 s. 0$\frac{3}{4}$ d.
	£5	2	11$\frac{1}{4}$	cost, at......... 3 s. 9$\frac{3}{4}$ d.

Again, without barring, thus:

	£	s.	d.
3 s. 4 d. is $\frac{1}{6}$ of £1...	27		
4 d. is $\frac{1}{10}$ of 3 s. 4 d.	4	10	0
1 d. is $\frac{1}{4}$ of 4 d......	0	9	0
2 qrs. is $\frac{1}{2}$ of 1 d......	0	2	3
1 qr. is $\frac{1}{2}$ of 2 qrs...	0	1	1$\frac{1}{2}$
	0	0	6$\frac{3}{4}$
	£5	2	11$\frac{1}{4}$

Assuming 27 s. for the multiplicand, we multiply by 3 s., which are similar units, (332,) and take parts of 1 s. for 9$\frac{3}{4}$d., thus:

		s.	
6 d. is ½ s.......		27	
		3 s.	9¾ d.
	£4	1 s.	
3 d. is ½ of 6 d.	0	13	6 d.
¾ is ¼ of 3 d.		6	9
		1	8¼
	£5	2	11¼

Or we may proceed thus:

	£	s.	
	1	7	
		3 s.	9¾ d.
9 d. is ¼ of 3 s.	4	1	
¾ d. is $\frac{1}{12}$ of 9 d.	1	0	3 d.
	0	1	8¼
	£5	2	11¼

From the above, the student will see that the method pursued may be greatly varied, according to the caprice of the operator; also, that the best methods to practise, practice alone will suggest.

PRACTICE TABLES.

Aliquot parts of a Dollar.		*Aliquot parts of a Shilling.*	
cts.	$	d.	s.
50	½	6	½
33⅓	⅓	4	⅓
25	¼	3	¼
16⅔	⅙	2	⅙
12½	⅛	1½	⅛
6¼	$\frac{1}{16}$	1	$\frac{1}{12}$

Aliquot parts of a Pound.

s.	d.	£	s.	d.	£
10	0	½	1	4	$\frac{1}{15}$
6	8	⅓	1	3	$\frac{1}{16}$
5	0	¼	1	0	$\frac{1}{20}$
4	0	⅕	0	8	$\frac{1}{30}$
3	4	⅙	0	6	$\frac{1}{40}$
2	6	⅛	0	4	$\frac{1}{60}$
2	0	$\frac{1}{10}$	0	3	$\frac{1}{80}$
1	8	$\frac{1}{12}$	0	2	$\frac{1}{120}$

Aliquot parts of a Cwt. lb.	cwt.
56	$\frac{1}{2}$
28	$\frac{1}{4}$
16	$\frac{1}{7}$
14	$\frac{1}{8}$
8	$\frac{1}{14}$
7	$\frac{1}{16}$

Aliquot parts of a Ton. cwt.	qr.	lb.	T.
5	0	0	$\frac{1}{4}$
4	0	0	$\frac{1}{5}$
2	3	12	$\frac{1}{7}$
2	2	0	$\frac{1}{8}$
1	1	20	$\frac{1}{14}$
1	1	0	$\frac{1}{16}$

Examples.

						£	s.	d.
1.	11 yds.	at	18 s.	8 d.	per yd.	10	5	4
2.	12 yds.	"	2	6	"	1	10	0
3.	15 yds.	"	1	4	"	1	0	0
4.	16 yds.	"	1	3	"	1	0	0
5.	18 yds.	"	6	8	"	6	0	0
6.	17 yds.	"	1	9	"	1	9	9
7.	19 yds.	"	4	6	"	4	5	6
8.	27 yds.	"	11	9½	"	15	18	4½
9.	25 yds.	"	13	11¾	"	17	19	5¾
10.	33 yds.	"	17	6	"	28	17	6
11.	43 yds.	" £1	19	11¾	"	85	19	1¼
12.	135 yds.	" 1	18	8	"	261	0	0

13.	11 cwt.	3 qr.	14 lb.	at	$19	per T.	$11,28$\frac{1}{8}$
14.	5	1	12	"	56	"	15,00
15.	7	3	12	"	61,50	"	24,16$\frac{1}{14}$
16.	9	1	4	"	75,80	"	35,19$\frac{2}{7}$
17.	19	3	14	"	90	"	89,43$\frac{3}{4}$
18.	17	2	0	"	87,50	"	76,56$\frac{1}{4}$
19.	15	0	20	"	69	"	52,36$\frac{17}{28}$
20.	13	0	27	"	49,75	"	32,93$\frac{321}{448}$

It is usual in business *to take the nearest cent, and to reject fractions of a cent;* thus, in this last example, we should call the amount 32 dollars and 94 cents. The student, however, should not, in any case, be satisfied till he has obtained the exact answer.

334. *When the quantity is compound, and the value of its principal unit also compound, we find,* as in the first 12 examples of the preceding article, *the value of the given number of principal units, and then,* as in the last 8 examples, *take parts of the value of the principal unit for the lower denominations.*

To find the value of 11 cwt. 1 qr. 14 lb., at £3 15 s. 6 d. per cwt., we proceed thus:

	£	s.	d.	
1 qr. is ¼ cwt.	3	15	6	
5 s. is ¼ £....	11 cwt. 1 qr. 14 lb.			
	33			value of 11 cwt. at £3
	2	15		prod. for 15 s., or 3 times 5 s.; *i. e.* value of 11 cwt. at 15 s.
	2	15		
6 d. is $\frac{1}{10}$ of 5 s.	2	15		
	0	5	6	value of 11 cwt. at 6 d.
14 lb. is ½ qr.	0	18	10½	value of 1 qr.
	0	9	5¼	value of 14 lb.
	£42	18	9¾	

We first multiply £3 15 s. 6 d. by 11; that is, having multiplied 3 by 11, we take parts of 11, considered as pounds, for the 15 s. 6 d. Having thus taken 11 times £3 15 s. 6 d., the value of 1 cwt., we have the value of 11 cwt. Then, to have the value of 1 qr., we take ¼ of £3 15 s. 6 d., which gives 18 s. 10½ d. Lastly, to have the value of 14 lb., or ½ qr., we take ½ of 18 s. 10½ d., the value of 1 qr. Having added the several products, we have £42 18 s. 9¾ d. for the value of the whole, as required. Or thus:

	£	s.	d.	
1 qr. is ¼	3	15	6	
			11 cwt.	1 qr. 14 lb.
	41	10	6	
14 lb. is ½		18	10½	
		9	5¼	
	£42	18	9¾	

We here multiply £3 15 s. 6 d. by 11, as in art. 321; we then take parts for 1 qr. 14 lb., and having added, we have £42 18 s. 9¾ d. for the whole value, as before.

Proof.—By Fractions.

15 s. 6 d. $= 15\frac{1}{2}$ s. $= \frac{31}{2}$ s. $= £\frac{31}{40}$, and $£3\frac{31}{40} = £\frac{151}{40}$. Again, 1 qr. 14 lb. $= 1\frac{1}{2}$ qr. $= \frac{3}{2}$ qr. $= \frac{3}{8}$ cwt., and $11\frac{3}{8}$ cwt. $= \frac{91}{8}$ cwt. Then, $\frac{91}{8} \times \frac{151}{40} = \frac{13741}{320} =$ £42 18 s. 9¾ d.

By Decimals.

15 s. 6 d. = 15,5 s. = £,775; therefore, £3 15 s. 6 d. = £3,775. Again, 1 qr. 14 lb. = 1,5 qr. = ,375 cwt.; therefore 11 cwt. 1 qr. 14 lb. = 11,375 cwt. Then, 11,375 × 3,775 = £42,940625, and reducing the decimal, we have £42 18 s. 9¾ d.

This method often requires a great many figures, and is therefore very tedious; the scholar may, however, sometimes use it as proof.

Examples.

1.	4 cwt.	3qr.	14lb.	at	£2	10s.	6d.	per cwt.	£12	6s.	$2\frac{1}{4}$ d.
2.	7	0	19	"	3	14	$8\frac{1}{2}$	"	26	15	$7\frac{1}{2}$
3.	13	2	7	"	2	3	$9\frac{3}{4}$	"	29	14	$2\frac{1}{2}$
4.	9	1	12	"	5	11	$6\frac{3}{4}$	"	52	3	$10\frac{3}{4}$
5.	29	3	17	"	7	1	$4\frac{1}{2}$	"	211	7	$4\frac{1}{4}$

6. What is the cost of 297 T. 13 cwt. 2 qr. 7 lb., at £13 12 s. $0\frac{1}{2}$ d. per ton? Ans. £4049 0 s. $10\frac{153}{640}$ d.

7. How much work will £74 2 s. $9\frac{1}{11}$ d. pay for, at the rate of 11 square yards for £1? Ans. 815 sq. yds. $4\frac{13}{20}$ sq. f.

8. If we pay £1 for 5 cwt. 2 qr. $3\frac{1}{2}$ lb. of potatoes, what quantity should we have for £63 10 s. 2 d.?

Ans. 17 T. 11 cwt. 1 qr. 3 lb. $6\frac{3}{5}$ oz.

9. What is the cost of 91 acres at £15 10 s. $1\frac{1}{2}$ d. per acre? Ans. £1411 1 s. $4\frac{1}{2}$ d.

10. How much land has the man who pays £15 10 s. $1\frac{1}{2}$ d. tax, at the rate of £1 for every 91 Acres?

Ans. 1411 A. 0 R. 11 P.

Calculation by Complements.

335. As a general discussion of this subject would perhaps occupy more pages than are contained in this entire work, we shall here chiefly consider the advantages it affords in Practice calculations.

When a quantity is multiplied by any proper fraction, $\frac{n}{m}$, it is diminished *by such part of itself as is expressed by the complement of that fraction*, that is, by $\frac{m-n}{m}$*ths* of itself. Thus:

$$Q \times \frac{n}{m} = \frac{nQ}{m}. \quad \text{Also, } Q - \frac{Q(m-n)}{m} = \frac{mQ - mQ + nQ}{m} = \frac{nQ}{m}. \quad (135.)$$

If we put $Q = 36$, and $\frac{n}{m} = \frac{7}{9}$, we shall have $36 \times \frac{7}{9} = 36 - \frac{2}{9}$ of $36 = 36 - 8 = 28$.

336. When $m - n = 1$, that is, when the numerator is only one unit less than the denominator, the above property

affords great advantage in calculation, seeing that *we have only to subtract* $\frac{1}{m}$*th.* For example, if we would multiply 3848 by $\frac{7}{8}$, as the complement of $\frac{7}{8}$ is $\frac{1}{8}$, we have only to subtract from 3848 its eighth part, thus: 3848 — 481 = 3367, which is easier than the following:

$$\begin{array}{r} 3848 \\ 7 \\ \hline 8)\,26936 \\ \hline 3367 \end{array}$$

Again, if we would find the cost of 3848 yds. at 17 s. 6 d. per yd., as 17 s. 6 d. is £$\frac{7}{8}$, we subtract from 3848 its eighth, as above, and have £3367 for the required cost. This is much easier than the following:

$$\begin{array}{l|r} & \pounds \\ 10\text{ s. is }\frac{1}{2}\ldots\ldots & 3848 \\ \cline{2-2} 5\text{ s. is }\frac{1}{2}\ldots\ldots & 1924 \\ 2\text{ s. }6\text{ d. is }\frac{1}{2} & 962 \\ & 481 \\ \cline{2-2} & \pounds 3367 \end{array}$$

What is the cost of 97 yds. at 83⅓ cts. per yd.?

$$\begin{array}{lr} & \$ \\ \text{Comp. of } 83\tfrac{1}{3} = 16\tfrac{2}{3} \text{ is } \tfrac{1}{6} & 97 \\ \quad\text{subtracted} & 16{,}16\tfrac{2}{3} \\ \cline{2-2} & 80{,}83\tfrac{1}{3} \end{array}$$

Here 97 yds. at \$1, would be \$97. Then as 83⅓ is $\frac{5}{6}$ of a dollar, and its comp. $\frac{1}{6}$, we subtract from 97 its sixth part, and have \$80,83⅓ for the required cost.

This is certainly more expeditious than the following:

$$\begin{array}{rl} \text{cts.} & \\ 83\tfrac{1}{3} & \\ 12 & \times 8 + 1 = 97 \\ \hline 1000 & \\ 8 & \\ \hline 8000 & \\ 83\tfrac{1}{3} & \\ \hline \$80{,}83\tfrac{1}{3} & \end{array}$$

As a mere introduction to the use which the intelligent student may hereafter make of this principle, we here insert a few

Tables of Complements.

cts.	£	Comp.
$93\frac{3}{4}$	$\frac{15}{16}$	$\frac{1}{16}$
$87\frac{1}{2}$	$\frac{7}{8}$	$\frac{1}{8}$
$83\frac{1}{3}$	$\frac{5}{6}$	$\frac{1}{6}$
75	$\frac{3}{4}$	$\frac{1}{4}$
$66\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

d.	s.	Comp.
11	$\frac{11}{12}$	$\frac{1}{12}$
$10\frac{1}{2}$	$\frac{7}{8}$	$\frac{1}{8}$
10	$\frac{5}{6}$	$\frac{1}{6}$
9	$\frac{3}{4}$	$\frac{1}{4}$
8	$\frac{2}{3}$	$\frac{1}{3}$

s.	d.	£	Comp.
19	11	$\frac{239}{240}$	$\frac{1}{240}$
19	10	$\frac{119}{120}$	$\frac{1}{120}$
19	9	$\frac{79}{80}$	$\frac{1}{80}$
19	8	$\frac{59}{60}$	$\frac{1}{60}$
19	6	$\frac{39}{40}$	$\frac{1}{40}$
19	4	$\frac{29}{30}$	$\frac{1}{30}$
19	0	$\frac{19}{20}$	$\frac{1}{20}$
18	9	$\frac{15}{16}$	$\frac{1}{16}$

s.	d.	£	Comp.
18	8	$\frac{14}{15}$	$\frac{1}{15}$
18	4	$\frac{11}{12}$	$\frac{1}{12}$
18	0	$\frac{9}{10}$	$\frac{1}{10}$
17	6	$\frac{7}{8}$	$\frac{1}{8}$
16	8	$\frac{5}{6}$	$\frac{1}{6}$
16	0	$\frac{4}{5}$	$\frac{1}{5}$
15	0	$\frac{3}{4}$	$\frac{1}{4}$
13	4	$\frac{2}{3}$	$\frac{1}{3}$

qr.	lb.	cwt.	Comp.
3	24	$\frac{27}{28}$	$\frac{1}{28}$
3	21	$\frac{15}{16}$	$\frac{1}{16}$
3	20	$\frac{13}{14}$	$\frac{1}{14}$
3	14	$\frac{7}{8}$	$\frac{1}{8}$
3	12	$\frac{6}{7}$	$\frac{1}{7}$
3	0	$\frac{3}{4}$	$\frac{1}{4}$

cwt.	qr.	lb.	T.	Comp.
18	3	0	$\frac{15}{16}$	$\frac{1}{16}$
18	2	8	$\frac{13}{14}$	$\frac{1}{14}$
17	2	0	$\frac{7}{8}$	$\frac{1}{8}$
17	0	16	$\frac{6}{7}$	$\frac{1}{7}$
16	0	0	$\frac{4}{5}$	$\frac{1}{5}$
15	0	0	$\frac{3}{4}$	$\frac{1}{4}$

Examples.

1.	16 yds.	at	$93\frac{3}{4}$ cts.	$15,00		
2.	17	"	$87\frac{1}{2}$	$14,87\frac{1}{2}$		
3.	19	"	$83\frac{1}{3}$	$15,83\frac{1}{3}$		
4.	25	"	75	18,75		
5.	37	"	$66\frac{2}{3}$	$24,66\frac{2}{3}$		
6.	45	"	11 d.	£2	1 s.	3 d.
7.	$55\frac{1}{2}$	"	$10\frac{1}{2}$	2	8	$6\frac{3}{4}$
8.	$96\frac{1}{4}$	"	10	4	0	$2\frac{1}{2}$
9.	$18\frac{3}{4}$	"	9	0	14	$0\frac{3}{4}$
10.	$87\frac{3}{4}$	"	8	2	18	6
11.	100	"	19 s. 9 d.	98	15	0
12.	120	"	19 8	118	0	0
13.	45	"	19 6	43	17	6
14.	16	"	18 9	15	0	0
15.	5	"	16 8	4	3	4

16. 3 qr. 24 lb. at \$19 per cwt. . . . . $\$18,32\frac{1}{7}$
17. 3 21 " 18. . . . . $16,87\frac{1}{2}$
18. 3 20 " 35. 32,50
19. 3 14 " 29. . . . . $25,37\frac{1}{2}$
20. 3 12 " 31,50. 27,00
21. 18 cwt. 3 qr. 0 lb. at £3 12 s. per T. £3 7 s. 6 d.
22. 18 2 8 " 4 10 " 4 3 $6\frac{6}{7}$
23. 17 2 0 " 5 6 8 d. " 4 13 4
24. 17 0 16 " 23 11 11 " 20 4 6

337. Let Q be any quantity, $\frac{n}{m}$ an improper fraction, and $n - m = d$. Then (77) $n = m + d$: consequently $\frac{nQ}{m} = \frac{mQ + dQ}{m} = Q + \frac{dQ}{m}$; that is, Q is increased in a ratio expressed by the fraction $\frac{n-m}{m}$, which is *the supplement of* $\frac{n}{m}$ *or part by which* $\frac{n}{m}$ exceeds a unit.

338. When $n - m = 1$, *the quantity* Q *is increased by* $\frac{1}{m}$ *th of itself;* that is, $\frac{nQ}{m} = Q + \frac{Q}{m}$. Thus $24 \times \frac{9}{8} = 24 + 3 = 27$.

Also, 24 yds. at $\$1,12\frac{1}{2}$ per yd. = \$24 + \$3 = \$27.

Here we have added to \$24, the cost of 24 yds. at \$1, the cost of 24 yds. at $\$\frac{1}{8}$, which is \$3.

Proof.

24
$112\frac{1}{2}$ cts.
2688
12
\$27,00

Required the cost of $29\frac{1}{4}$ yds. at $\$1,33\frac{1}{3}$ per yd.

Here the supplement is $33\frac{1}{3}$ cts. = $\$\frac{1}{3}$. We therefore add to $29\frac{1}{4}$, considered as $\$29\frac{1}{4}$, its third part, thus: $29\frac{1}{4} + 9\frac{3}{4} =$ \$39, Answer.

Examples.

1. $18\frac{3}{4}$ yds. at $\$1,33\frac{1}{3}$. \$25,
2. $25\frac{1}{2}$ " 1,25. . . . . $31,87\frac{1}{2}$

3.	$37\frac{3}{4}$ yds.	at	$1,16\frac{2}{3}$	\$$44,04\frac{1}{6}$
4.	43	"	$1,12\frac{1}{2}$	$48,37\frac{1}{2}$
5.	64	"	$1,06\frac{1}{4}$	68,00
6.	125	"	$1,66\frac{2}{3}$	$208,33\frac{1}{3}$

Here, as it is of no consequence which number we take for multiplicand, we add to \$$1,66\frac{2}{3}$ its fourth part, which is $41\frac{2}{3}$, and have $1,66\frac{2}{3} + 41\frac{2}{3} =$ \$$208\frac{1}{3}$ or \$$208,33\frac{1}{3}$, for the answer.

Proof. $\frac{1,66\frac{2}{3} \times 1000}{8} = \frac{1666,66\frac{2}{3}}{8} = 208,33\frac{1}{3}$. See art. 273.

SECTION XVIII.

DUODECIMALS—BOARD MEASURE, ETC.

Duodecimals.

339. This rule is used in measuring cord-wood, artificer's work, &c. The foot is the principal measure, and is divided as follows:

1 foot, *ft.* is equal to	12 inches, or primes, marked thus,	′	$(\frac{1}{12})$
1 inch	12 parts, or seconds	″	$(\frac{1}{12})^2$
1 second	12 thirds	‴	$(\frac{1}{12})^3$
1 third,	12 fourths	⁗	$(\frac{1}{12})^4$

The inch, or prime, is $\frac{1}{12}$ of a foot. The second is $(\frac{1}{12})^2$, or $\frac{1}{12}$ of $\frac{1}{12} = \frac{1}{144}$ of a foot. The third is $(\frac{1}{12})^3$, or $\frac{1}{12}$ of $\frac{1}{144} = \frac{1}{1728}$ of a foot, &c.

These parts, from their regular ratio of 12, are called *duodecimals*, which name is derived from the Latin *duodecim*, twelve.

Multiplication of Duodecimals.

340. From the nature of fractions, we shall easily perceive what parts will be produced in multiplying together any of the above denominations. For example, if we multiply by feet, as these are integers (whole numbers,) having a unit for denominator, it is evident that *the denominator of the part multiplied will not be altered;* thus, 3 feet multiplied by 2 feet, will give 6

feet; 3 in. multiplied by 2 *ft.*, that is, $\frac{3}{12} \times \frac{2}{1} = \frac{6}{12}$, or 6 *in.* Again, 3 *sec.* multiplied by 2 *ft.*, or $\frac{3}{144} \times \frac{2}{1} = \frac{6}{144}$, or 6 seconds, etc.

If we multiply by inches, as the denominator of these is 12, *the denominator of the part multiplied will become the same as that of the next lower denomination;* thus, 3 ft. multiplied by 2 in. will give 6 in.; that is, $\frac{3}{1} \times \frac{2}{12} = \frac{6}{12}$, or 6 inches; 3 in. multiplied by 2 in.; that is, $\frac{3}{12} \times \frac{2}{12} = \frac{6}{144}$, or 6 seconds; 3 sec. multiplied by 2 in., or $\frac{3}{144} \times \frac{2}{12} = \frac{6}{1728}$, or 6 thirds, &c. From what has been said, the products of any other parts will easily be estimated by one who has learned fractions.

341. As the second is the square of the prime, the third, the cube, &c., if we consider the feet as having 0 for their index, the inches 1, the seconds 2, the thirds, 3, &c., if we multiply these together, the parts produced will have for their index *the sum of the indices of the parts multiplied;* thus, the index of the prime is 1; that of the second is 2, and the sum 3, of these indices, is the index of the parts produced when we multiply inches by seconds. It is the same for any other parts, seeing that *their product contains the factor $\frac{1}{12}$, as often as its index contains a unit.*

For example, to multiply 7 ft. 3 in. 4 sec. by 2 ft. 4 in., we write the numbers thus:

$$\begin{array}{rrrr}
7^{\circ} & 3' & 4'' & \\
2^{\circ} & 4' & & \\
\hline
14 & 6 & 8 & \\
2 & 5 & 1 & 4''' \\
\hline
16 & 11 & 9 & 4
\end{array}$$

and first multiply all the parts of the multiplicand by the 2 ft. of the multiplier. Beginning with the lowest denomination, which is here seconds, we say $2^{\circ} \times 4'' = 8''$, which we place under seconds; then $2^{\circ} \times 3' = 6'$, which we write under inches: lastly, $2^{\circ} \times 7^{\circ} = 14^{\circ}$, which we write under feet.

Then, to multiply by $4'$, we say $4' \times 4'' = 16''' = 1'' + 4'''$; we therefore write $4'''$ one place farther to the right, and retain the $1''$, to add to the next product. Then $4' \times 3' = 12''$, and, adding the $1''$ retained, we have $13'' = 1' + 1''$; we therefore write $1''$ under $8''$, and retain the $1'$ to add to the next product. Lastly, $4' \times 7^{\circ} = 28'$, and $1'$ retained, is $29' = 2^{\circ}$ $5'$; we therefore write $5'$ under the primes or inches, and the

2 ft. under feet. Having added the two products, we have 16 feet, 11 primes, 9 seconds, 4 thirds, for the total product.

342. We have seen (110) that the content of any superficies is estimated in squares, and is the product of the length and breadth, each taken in the same measure. Artificers' work, such as glazing, painting, plastering, paving, roofing, &c., is usually paid by the square yard or square foot; and, as the dimensions are taken in feet and inches, being generally measured with a carpenter's rule, such work is conveniently calculated by duodecimals. Parts less than seconds are commonly *omitted in practice.*

Suppose the roof of a green-house to be 100 ft. 9 in. long and 21 ft. 6 in. wide, what is the amount of glazing, and what will it cost at 9 cts. per square foot?

$$\begin{array}{rrr}
100^\circ & 9' & \\
21 & 6 & \\
\hline
2115 & 9 & \\
50 & 4 & 6'' \\
\hline
2166^\circ & 1' & 6''
\end{array}$$

The scholar must take particular notice, that in the product, the prime 1′ is $\frac{1}{12}$ of a square foot, or of 144 square inches, and is consequently 12 *square inches;* that the 6″, being a number of hundred and forty-fourths, is 6 *square inches,* and hence, that $1' \, 6'' = \frac{12}{144} + \frac{6}{144} = \frac{18}{144} = \frac{1}{8}$; or, recollecting that $1\frac{1}{2}$ d. $= \frac{1}{8}$ of a shilling, he will see that 1′ 6″ or $1\frac{1}{2}$ twelfths of a s. f. $= \frac{1}{8}$ of a s. f. Therefore, 2166° 1′ 6″ $= 2166\frac{1}{8}$ s. f.; which, as dollars and cents are decimals, we may express thus: 2166,125 s. f. Then, multiplying by 9, we have \$194,95125 or \$194,95$\frac{1}{8}$ for the required cost.

Examples.

1. Suppose a room to be 47 ft. 2 in. long, and 13 ft. 6 in. wide; what will the ceiling of it cost, at 25 cts. per square yard? Ans. \$17,68$\frac{3}{4}$

2. How many yards of carpeting, that is yard-wide, will carpet a room that is 36 ft. 8 in. long, and 25 ft. 6 in. wide, and what will it cost, at \$1,12$\frac{1}{2}$ cts per yard?
Ans. 103$\frac{8}{9}$ yds., and the cost \$116,87$\frac{1}{2}$.

3. How many square yards of paving are there in a court-yard, which is 77 ft. 3 in. long, and 66 ft. 8 in. wide; how many bricks will pave it, allowing each brick, when laid, to cover a

space of 8 in. long and 4 in. wide; and how much will the paving of it cost at 56$\frac{1}{4}$ cts. per square yard?

Ans. 572$\frac{2}{9}$ s. yds.; 23175 bricks, and the cost $321,87$\frac{1}{2}$.

343. Duodecimals may also be applied to quantities of three dimensions, the content or capacity of which, as we have seen (170,) is estimated in cubes, and is found by multiplying together their length, breadth, and thickness.

Required, the content, in bushels, of a granary bin, the length being 53 ft. 9 in.; the breadth 6 ft. 3 in. and the depth 4 ft. 5 in.

	53°	9′			
	6°	3′			
	322	6			
	13	5	3″		
	335	11	3		square feet and parts.
	4	5			
	1343	9	0		
	139	11	8	3‴	
	1483°	8′	8″	3‴	content of bin in c. f.
Subtract $\frac{1}{5}$	296	8	11	3	see art. 288.
	1186	11	9	0	content in bushels, $\frac{1}{215}$
Add $\frac{1}{215}$	5	6	3	0	too little (288)
	1192	6	0	0	true content in bushels.

The scholar must here observe with attention, that in the product 1483° 8′ 8″ 3‴, the number 1483 is a number of cubic feet (170); also, that each unit in the 8′, being $\frac{1}{12}$ of a cubic foot, or $\frac{1}{12}$ of 1728 cubic inches, is 144 cubic inches; each unit in the 8″, being $\frac{1}{12}$ of 1′, or $\frac{1}{12}$ of 144 cubic inches, is 12 cubic inches; and lastly, that each unit in the 3‴, being $\frac{1}{12}$ of 1′, or $\frac{1}{12}$ of 12 cubic inches, is 1 cubic inch. We can, therefore, reduce these parts to cubic inches, thus: $8' + 8'' + 3''' = (8 \times 144) + (8 \times 12) + 3 = 1152 + 96 + 3 = 1251$ cubic inches, or $\frac{1251}{1728} = \frac{139}{192}$ of a cubic foot.

Therefore, 1483° 8′ 8″ 3‴ = 1483$\frac{139}{192}$ cubic feet.

Having reduced the cubic feet to bushels, by the method given in art. 288, we have 1192° 6′, or 1192$\frac{1}{2}$ bush. for the required content.

Proof by Fractions.

$53° 9' = 53\frac{3}{4} = \frac{215}{4}$; $6° 3' = 6\frac{1}{4} = \frac{25}{4}$; $4° 5' = 4\frac{5}{12} = \frac{53}{12}$; then, subtracting $\frac{1}{5}$ from one of the factors, namely the factor $\frac{25}{4}$, which makes it $\frac{20}{4} = 5$, and adding $\frac{1}{215}$ to one of the factors, namely to $\frac{215}{4}$, which makes it $\frac{216}{4} = 54$; we have, $\frac{53}{12} \times \frac{54}{1} \times \frac{5}{1} = \frac{53}{2} \times \frac{9}{1} \times \frac{5}{1} = \frac{2385}{2} = 1192\frac{1}{2}$ for the number of bushels.

Parts less than thirds are *not noticed* in common practice.

Examples.

1. What is the content of a cave, the length of which is 12 ft. 9 in.; the breadth 7 ft. 6 in., and the height 8 ft. 3 in.?

Ans. 788 c. f. 10′ 10″ 6‴, or $788\frac{29}{32}$ c. f.

2. Suppose that the body of a cart is 7 ft. 2 in. long, inside, 3 ft. 6 in. wide, and 1 ft. 1 in. deep; how many times can the cart be filled from a pit that is to be dug in the earth, 45 ft. 3 in. long, 10 ft. 2 in. wide, and 6 ft. 1 in. deep; allowing the earth in the pit and in the cart to be equally compact; and how many cubic yards of earth does said pit contain?

Ans. The cart can be filled $102\frac{7741}{7826}$, or nearly 103 times, and the pit contains $103\frac{5065}{7776}$, or not quite 104 cubic yards.

344. The transportation of merchandise in ships is called freight, and the freight of all irregular packages, such as bales, boxes, &c. is charged by the cubic foot; the parts less than half a cubic foot being neglected, and those amounting to half a cubic foot or more accounted a cubic foot each; that is to say, *the content of each package is calculated to the nearest cubic foot.* When the packages are uniform, the freight is charged by the barrel, bag, &c.

Examples.

Required the content of the following bales, boxes, or packages, the length, breadth, and depth being given in feet and inches:

1. 5 ft. 3 in.; 3 ft. 1 in.; 2 ft. 9 in. Content, 45 c. f.
2. 6 ft. 2 in.; 3 ft. 4 in.; 3 ft. 1 in. " 63 "
3. 7 ft. 5 in.; 4 ft. 9 in.; 3 ft. 6 in. " 123 "
4. 8 ft. 1 in.; 4 ft. 10 in.; 3 ft. 8 in. " 143 "
5. 9 ft. 4 in.; 6 ft. 5 in.; 5 ft. 7 in. " 334 "

6. Suppose the bushel to contain exactly 2150 cubic inches, and that the inside of a box is the exact cube of 5 feet; how

much must the edge of the box be planed down, that it may contain just 100 bushels?

Ans. $\frac{5}{18}$, or not quite $\frac{1}{3}$ of an inch.

To Measure Cord-wood.

345. The cord by which wood is measured is 128 c. f., or the content of a pile of wood 8 feet long, 4 ft. wide, and 4 ft. high.

The content, therefore, of any quantity of cord-wood, is calculated by comparing it with a pile of the above dimensions. Hence, the rule, *multiply the length, breadth, and height together, all in feet, and divide by* 128, which will give the content in cords. Or rather, set down, in continued multiplication, the length, breadth, and height, as a numerator; and 8 × 4 × 4, the dimensions of a cord, as a denominator, and find the result as in multiplication of fractions.

Required the content, in cords, of a pile of wood that is 60 ft. long, 16 ft. wide and 10 ft. high.

$$\frac{16 \times 60 \times 10}{4 \times 4 \times 8} = 15 \times 5 = 75 \text{ cords. Answer.}$$

A cord is sometimes said to contain 8 feet of wood, each foot being consequently, 16 cubic feet. Now, on a base 8 ft. long and 4 ft. wide, it is plain that every 6 inches in height will give 1 foot of wood. For, $8 \times 4 \times \frac{1}{2} = 16$ cubic feet.

346. When the dimensions are feet and inches, we may find the content by duodecimals, and divide by 128.

What is the content of a pile of wood, the length of which is 20 ft. 9 in., the breadth 3 ft. 7 in., and the height 9 ft. 8 in.?

$$\begin{array}{rrrl}
 & 20^{\circ} & 9' & \\
 & 9 & 8 & \\
\hline
 & 186 & 9 & \\
 & 13 & 10 & 0'' \\
\hline
 & 200 & 7 & \\
 & 3 & 7 & \\
\hline
 & 601 & 9 & \\
 & 117 & 0 & 1'' \\
\hline
128) & 718 & 9' & 1''\ (5 \text{ C. } 5 \text{ f., nearly.} \\
 & 640 & & \\
\hline
16) & 78\ (4\frac{7}{8} & & \\
 & 64 & & \\
\hline
 & \frac{14}{16} = \frac{7}{8} & &
\end{array}$$

We here divide by 128, for cords, and the remainder by 16, for feet of wood. As we only calculate to the nearest foot of wood, in dividing by 128, *we neglect the parts inferior to cubic feet*, which, as they never can equal $\frac{1}{16}$ of a foot of wood, cannot affect the result. In dividing by 16, as the fraction $\frac{7}{8}$ exceeds $\frac{1}{2}$, we add 1 to the feet, which gives for the content 5 cords, 5 feet, or $5\frac{5}{8}$ cords, nearly.

Examples.

1. What is the content of a pile of wood, 24 ft. long, 20 ft. broad, and 18 ft. high? Ans. $67\frac{1}{2}$ cords.

2. What is the content of a pile, 17 ft. 8 in. long, 9 ft. 4 in. broad, and 10 ft. 7 in high?
Ans. 13 C. 5 f., or $13\frac{5}{8}$ cords.

3. What is the content of a pile, the length being 101 ft., the breadth 17 ft., and the height 8 ft.?
Ans. 107 C. $2\frac{1}{2}$ ft., or $107\frac{5}{16}$ cords.

4. What is the content of a pile, the length being 203 ft., the breadth 14 ft. 9 in., and the height 7 ft. 6 in.?
Ans. 175 C. $3\frac{1}{2}$ ft., or $175\frac{7}{16}$ cords, nearly.

Board Measure, &c.

In this is included the measurement of planks, scantling, rafters, and hewn timber.

To measure boards, the usual thickness being one inch:

347. The length, in inches, multiplied by the breadth, in inches, will evidently give the number of square inches in the surface of the board. This divided by 144, the number of square inches in a square foot, will give the content of the surface of the board in square feet.

Put $l =$ the length of the board, in feet; then, $l \times 12 =$ the length, in inches; now put $w =$ the width, in inches, then $l \times w \times 12 =$ the content of surface, in square inches, and

$$\frac{l \times w \times 12}{144} \text{ or, } \frac{l \times w \times \cancel{12}}{12 \times \cancel{12}} = \frac{l \times w}{12} = \text{content in square feet.}$$

Now, if $l = 12$, that is, if the board is 12 feet long, then $\frac{l \times w}{12} = \frac{\cancel{12} \times w}{\cancel{12}} = w$; that is, *the number which shows the width in inches, is the number of square feet contained in the surface* of a board, when the board is twelve feet long.

Again, if $w = 12$, then $\frac{l \times w}{12} = \frac{l \times \cancel{12}}{\cancel{12}} = l$, or the length.

In other words, when the board is 12 in. wide, every foot in length will make a square foot on its surface.

From the above the following rules may be deduced:

1. For a board of any length: Find the content of its surface for 12 feet long; *i. e.* set down its width, in inches, as square feet; take parts for the surplus or deficient length, and add, or subtract, as occasion may require. Thus: for a board 14 ft. long and 8 in. wide, we set down 8 s. f. content for 12 ft. Then, 2 ft. being $\frac{1}{6}$ of 12, we take $\frac{1}{6}$ of $8 = 1\frac{1}{3}$; and $8 + 1\frac{1}{3} = 9\frac{1}{3}$ s. f. is the required content.

Had the board been only 10 ft. long, we should, by a similar reasoning, have had $8 - 1\frac{1}{3} = 6\frac{2}{3}$ s. f. for its content.

For a board 16 ft. long, the surplus 4 ft. being $\frac{1}{3}$ of 12, its width, in inches, taken as s. f. plus $\frac{1}{3}$ of the same, is its content.

For a board 18 ft. long, the surplus 6 ft. being $\frac{1}{2}$ of 12, its width, as before, plus $\frac{1}{2}$ its width, is its content; and so, for any other length, the parts may be easily taken.

2. For a board of any width: Set down its length, in feet, which is its content in s. f., when the width is 1 foot, or 12 inches. Then take parts for the surplus or deficient width, and add or subtract, as before.

What is the content of a board, 22 ft. long and 9 in. wide?

6 in. is $\frac{1}{2}$...	22	s. f., content for 1 ft. wide.
3 in. is $\frac{1}{2}$...	11	content for 6 in. wide.
	$5\frac{1}{2}$	content for 3 in. wide.
	$16\frac{1}{2}$	s. f. required content, or content for 9 inches.

Or, as the deficiency 3 in. is $\frac{1}{4}$ of a foot, we set down 22, and subtract its fourth part, thus:

3 in. is $\frac{1}{4}$...	22	s. f., content for 1 foot wide.
	$5\frac{1}{2}$	content for 3 in. wide.
	$16\frac{1}{2}$	difference, or required content.

Suppose the board 10 in. wide, we subtract $\frac{1}{6}$, thus:

2 is $\frac{1}{6}$...	22	
	$3\frac{2}{3}$	
	$18\frac{1}{3}$	s. f., content

Suppose it 7 in. wide, we take $\frac{1}{2}$ for 6 in. wide, and add $\frac{1}{6}$ of the quotient, for 1 in., thus:

$$\begin{array}{r|l} \text{6 in. is } \frac{1}{2} & 22 \\ \cline{2-2} \text{1 in. is } \frac{1}{6} & 11 \\ & 1\frac{5}{6} \\ \cline{2-2} & 12\frac{5}{6} \text{ or 13 s. f., content.} \end{array}$$

In a board-yard, the length of a board is always an even number of feet, or is so considered, and the content is calculated to the nearest square foot.

Examples.

1. Required the content of a pile of boards 12 ft. long, the several widths being as follows: 10 boards 8 in. wide; 14 boards 9 in.; 21 boards 12., and 7 boards 16 in. wide.
Ans. 570 s. f.

2. Required the content of a pile of boards 16 ft. long, the widths being as follows: 19 boards 9 in.; 25 boards 8 in.; 13 boards 11 in.; 8 boards 15 in., and 5 boards 17 in. wide.
Ans. 959 s. f.

3. Required the content of a pile of boards of the following dimensions; always understood that the first number is the length in feet, and the second the width in inches: 14 by 8; 16 by 9; 12 by 7; 14 by 13; 18 by 16; 20 by 13; 18 by 16; 10 by 7; 18 by 12; 14 by 11; 16 by 18, and 22 by 19.
Ans. 209 s. f.

To measure planks, the thickness being more than an inch:

248. If, without altering the length, we double, treble, &c. the width of a board, it is easy to see that *we shall double, treble, &c. its content.*

Also, allowance being made for the saw, it is plain that a plank, two, three, &c. inches thick, may be sawn into two, three, &c. boards, one inch thick, of the length and breadth of the plank; consequently, if we double, treble, &c. the thickness of a board or plank, *we shall double, treble, &c. its content.*

It is therefore evident, that if we increase the width or thickness of a board or plank, by one-half, one-third, &c. of said width, or thickness, *we shall increase the board by one-half, one-third, &c. of its content.* Hence the following convenient rule:

Find how many times 1 inch is contained in the thickness, and multiply the width by the quotient. Then proceed as if to measure a board of the length of the plank, and of the increased width, which will give the content.

Thus, if a plank be 18 ft. long, 8 in. wide, and $1\frac{1}{2}$ in. thick, we say $1\frac{1}{2}$ times 8 in. $= 8 + 4 = 12$ in.; we then calculate as for a board 18 ft. long and 12 in. wide, and have 18 s. f., board measure. Had the plank been $1\frac{3}{4}$ in. thick, we should have said, $8 \times 1\frac{3}{4} = 8 + 6 = 14$ in. increased width. Then,

2 in. is $\frac{1}{6}$..	18	s. f., content for 1 ft. wide.
	3	content for 2 in.
	21	s. f., content of plank.

Or thus: 6 is $\frac{1}{2}$..	14	s. f., content for 12 ft.
	7	content for 6 ft.
	21	content, as before.

Had the plank been $2\frac{3}{4}$ in. thick, we should have said, $8 \times 2\frac{3}{4} = 16 + 6 = 22$ in. increased width. Then,

6 ft. is $\frac{1}{2}$..	22	s. f., content for 12 ft.
	11	content for 6 ft.
	33	required content.

Or thus: 6 ft. is $\frac{1}{2}$ of 12 ft.	18	s. f., content for 1 ft. wide.	
4 is $\frac{1}{3}$ of 12 ft.	9	"	6 in. "
	6	"	4 in. "
	33	s. f., as before.	

Suppose a plank is 32 ft. long, 13 in. wide, and $3\frac{1}{4}$ in. thick, $13 \times 3\frac{1}{4} = 39 + 3\frac{1}{4} = 42\frac{1}{4}$ in., increased width.

4 ft. is $\frac{1}{3}$...	$42\frac{1}{4}$		s. f., content for 12 ft. long.
	3		
	$126\frac{3}{4}$ or	$\frac{9}{12}$	s. f., content for 36 ft. long.
	14	$\frac{1}{12}$	cont. for 4 ft. subtracted.
	$112\frac{8}{12}$ or $112\frac{2}{3}$ s. f.		

Or thus: 6 in. is $\frac{1}{2}$...	32	s. f., cont. for 1 ft. wide.	
	3		
	96	s. f., cont. for 3 ft. wide.	
$\frac{1}{4}$ in. is $\frac{1}{24}$ of 6 in...	16	"	6 in. "
$\frac{16}{24} =$	$\frac{2}{3}$	"	$\frac{1}{4}$ in. "
	$112\frac{2}{3}$	content, as before.	

Or thus: 1 in. is $\frac{1}{12}$..... | 32 s. f., cont. for 1 ft. wide.

$2\frac{2}{3}$ cont. for 1 in. "

$34\frac{2}{3}$ cont. for 13 in. "

$3\frac{1}{4}$

104

$8\frac{2}{3}$

$112\frac{2}{3}$ content.

349. As the length, breadth, and thickness are all factors of the content, if we increase or diminish any one of these in any ratio, *the content will be increased or diminished in the same ratio;* therefore, we may, if we please, *multiply the length or breadth by the thickness, as is most convenient.* As the length is always an even number, this will often be found very useful.

Example.

Required the content of a pile of planks, the length, breadth, and thickness of each being given in the order here named: 10 ft. by 13 in., by $1\frac{1}{2}$ in.; 12 ft. by 17 in., by $2\frac{1}{4}$ in.; 14 ft. by 12 in., by $1\frac{3}{4}$ in.; 18 ft. by 13 in., by $3\frac{1}{2}$ in.; 16 ft. by 11 in., by $2\frac{1}{2}$ in.; 30 ft. by 13 in., by $4\frac{1}{4}$ in.; 36 ft. by 17 in., by $3\frac{3}{4}$ in.; 20 ft. by 19 in., by $5\frac{1}{4}$ in. Answer, 679 s. f.

350. We have seen (347) that $\frac{l \times w}{12}$ is the content of the surface of a board in square feet, and (348) that, for the content of stuff thicker than 1 in., we must multiply this content by the thickness: *the product,* therefore, *of the length, breadth, and thickness, divided by* 12, or, which is the same thing, *the product of the width and thickness, multiplied by* $\frac{l}{12}$, *is the content in board measure;* that is to say, it is the number of square feet of surface that the stuff would cover, if sawn into boards 1 in. thick, allowing no waste in sawing. This will also appear evident thus: in dividing by 12, we may suppose the width to be reduced to feet; and, as the length is already feet, the product of these will give the content of the surface (107) in square feet; wherefore, (348,) this multiplied by the thickness, will give the content in board measure.

To Measure Scantling:

351. When the dimensions of width and thickness are frac-

tional, reduce these to fractions, and find their product in its lowest terms. Then multiply this product by $\frac{l}{12}$, reduced to its lowest terms, which will give the content.

Let a piece of scantling be 16 ft. long and $3\frac{1}{2}$ in. by $2\frac{1}{2}$, $3\frac{1}{2}=\frac{7}{2}$; $2\frac{1}{2}=\frac{5}{2}$: also $\frac{l}{12}=\frac{16}{12}=\frac{4}{3}$; then $\frac{7}{2}\times\frac{5}{2}\times\frac{4}{3}=\frac{35}{3}$ $=11\frac{2}{3}$ s. f. board measure.

352. When the area of the end is 12 square inches, *the length of the piece is its content in square feet, board measure.* Also, if the area of the end is a multiple of 12 square inches, *the content is the same multiple of the length.* This is evident, because the denominator of the factor $\frac{l}{12}$ will, in both cases, be cancelled.

Example.

Required the content of the following pieces of scantling, the length being given in feet, and the breadth and thickness in inches and fractions of an inch: 20 ft. 3 by 4; 18 feet $3\frac{1}{2}$ by 3; 26 ft. $3\frac{1}{4}$ by $2\frac{1}{4}$; 16 ft., 6 by 4; 28 ft., $5\frac{1}{4}$ by $4\frac{3}{4}$; 30 ft., 6 by 4; 24 ft., $5\frac{3}{4}$ by $3\frac{3}{4}$; 14 ft., $5\frac{1}{2}$ by $4\frac{1}{4}$.

Answer, 272 s. f. B. M.

To measure Rafters, the width being greater at one end than the other, but the thickness uniform:

Add the greater and less width together, and take half the sum for a mean width. Then, with this mean width, the uniform thickness and length, proceed as for scantling.

Let a rafter be 40 ft. long, 4 in. by 9 at one end, and 4 by 5 at the other. $9+5=14$, and $\frac{14}{2}=7$ mean width. Then, mean area $7\times 4\div 12$, or $\frac{28}{12}=\frac{7}{3}$; and, lastly, $\frac{40}{1}\times\frac{7}{3}=\frac{280}{3}=93\frac{1}{3}$ s. f. content.

Or thus: 28 s. f. = content for 12 ft.; then $28\times 3=84$ content for 36 ft. Now, as 4 is $\frac{1}{3}$ of 12, we have $\frac{28}{3}=9\frac{1}{3}$ content for 4 ft. Lastly, $84+9\frac{1}{3}=93\frac{1}{3}$ s. f., as before.

Example.

Required the content of the following rafters: 40 ft., 5 by 10 and 5 by 5; 28 ft. 5 by 9 and 5 by 4; 24 ft., $4\frac{1}{2}$ by 10 and $4\frac{1}{2}$ by 6; 50 ft, $5\frac{1}{2}$ by 10 and $5\frac{1}{2}$ by 5; 36 ft., 6 by 10 and 6 by 6; 48 ft., 4 by 10 and 4 by 6. Answer, 686 s. f. B. M.

To find the Content of Hewn Timber in Cubic Feet:

353. Put $q =$ the length, and a and b, the two dimensions of the end in inches. Then $q \times a \times b \times 12$ is the content in cubic inches; and, as 1728 or $12 \times 12 \times 12$ cubic inches make one cubic foot, it is plain that

$$\frac{q \times a \times b \times 12}{12 \times 12 \times 12} = \frac{q \times a \times b}{12 \times 12} = \text{content in c.f.}$$

Hence the rule: Multiply the length in feet by the area of the end in inches, and divide by 144. But it is much better always to write the numbers in the form $\frac{q \times a \times b}{12 \times 12}$ on account of cancelling.

When the area of the end is 144 square inches, *the length of the piece is its content in* c. f.

Let a piece of timber be 36 ft. long, and 16 in. by 12 on the end, then:

$$\frac{q \times a \times b}{12 \times 12} = \frac{36 \times 16 \times 12}{12 \times 12} = 3 \times 16 = 48 \text{ c.f., the content.}$$

Examples.

1. Required the content in cubic feet of the following pieces of timber: 20 ft., 12 by 12; 30 ft., 12 by 10; 18 ft., 10 by 9; 24 ft., 12 by 11; 36 ft., 14 by 10; 40 ft., 16 by 12; 32 ft., 18 by 16; 34 ft., 13 by 12; and the cost at \$2,12$\frac{1}{2}$ per ton of 50 c.f. Answer, 267 c. f., and the cost \$11,34$\frac{3}{4}$.

2. Required the content of the following pieces, namely: two pieces, each 40 ft., 16 by 14; 3 pieces, each 36 ft., 19 by 18; 4 pieces, each 20 ft., 12 by 11; 3 pieces, 28 ft., 16 by 14; and 2 pieces, each 18 ft., 12 by 10; and the cost at \$3 per ton. Answer, 12 t. 15 c. ft., and the cost \$36,90.

3. Required the cost of 100 pieces, each 20 ft., 10 by 9, at \$2$\frac{1}{2}$ per ton; 50 pieces, each 30 ft., 14 by 14, at \$3 per ton, and 75 pieces, each 18 ft, 12 by 12, at \$4,50 per ton.
Answer, \$806,50.

METRICAL OR DECIMAL SYSTEM.

(From Kelley's "Universal Cambist.")

The fundamental standard adopted in France for the metrical system of weights and measures, is a quadrant of the meridian; that is to say, the distance from the equator to the north pole. This quadrant is divided into ten millions of equal parts, and one of these parts or divisions is called the *Metre*, which is adopted as the unit of length, and from which, by decimal multiplication and division, all other measures are derived. The length of the quadrant has been ascertained by MM. Delambre and Mechain, by measuring an arc of the meridian between the parallels of Dunkirk and Barcelona, and has been found to contain 5130740 French toises. This number, divided by ten millions, gives 36,941328 French inches, which is the *metre*, the element of all the other measures, and which is equal to 39,371 English inches.

In order to express the decimal proportions, the following vocabulary of names has been adopted:

For multipliers, the word	*Deca* prefixed means	10	times.
" "	*Hecto*	100	"
" "	*Kilo*	1000	"
" "	*Myria*	10,000	"
On the contrary, for divisors, the word	*Deci* expresses the	10th	part.
" "	*Centi* ..	100th	"
" "	*Milli* ...	1000th	"

It may assist the memory to observe that the terms for multiplying are Greek, and those for dividing, Latin.

Thus, *Deca-metre*, means ten metres.
Deci-metre, the 10th part of a metre.
Hecta-metre 100 metres.
Centi-metre, the 100th part of a metre, and so on for the rest.

The *metre*, as before stated, is the element of long measure, and = 39,371 English inches.

The are, which is a square *deca-metre*, (or 100 square metres,) is the element of superficial measures. It equals 3,955 English perches.

The *stere*, which is a cubic *metre*, is the element of solid measures, and = 35,317 cubic feet English.

The *Litre*, which is the cubic *deci-metre*, is the element of all liquid measures, and of all other measures of capacity. It equals 0,26419 English gallons, and the hectolitre = 2,8379 Winchester bushels.

Lastly, the *gramme*, which is the weight of a cubic *centimetre* of distilled water, of the temperature of melting ice, (the greatest condensation,) is the element of all weights, and equals 15,434 English grains, Troy.

New Weights and Measures of France, compared with the old, and also with English Weights and Measures.

LINEAR MEASURE.

	French Feet.	English Feet.
Distance from the equator to the pole	30784440	32809167
Degree (centesimal)	307844,4	328091,67
Myriametre	30784,44	32809,167
Kilometre	3078,444	3280,9167
Hectometre	307,8444	328,09167
Decametre	30,78444	32,809167
Metre	3,078444	3,2809167

	French Lines.	English Lines.
Decimetre	44,3296	47,2452
Centimetre	4,43296	4,72452
Millimetre	0,443296	,472452

SQUARE OR SUPERFICIAL MEASURE.

	French Square Feet.	English Square Feet.
Myriare	9476817,46113	10764414,3923
Kilare	947681,746113	1076441,43923
Hectare	94768,1746113	107644,143923
Decare	9476,81746113	10764,4143923
Are	947,681746113	1076,44143923
Deciare	94,7681746113	107,644143923
Centiare (metre carré)	9,47681746113	10,7644143923

	French Square Inches.	English Square Inches.
Square decimetre	13,646617	15,500756

	French Square Lines.	English Square Lines.
Square centimetre	19,651184	22,321088
Square millimetre	0,19651184	0,2232108

MEASURES OF CAPACITY.

	French Cubic Feet.	English Cubic Feet.
Myrialitre	291,738519	353,1714695
Kilolitre (metre cube)	29,1738519	35,31714695
Hectolitre	2,91738519	3,5817146945

	French Cubic Inches.	English Cubic Inches.
Decalitre	504,124160	610,2802806
Litre (decimetre cube)	50,4124160	61,02802806
Decilitre	5,04124160	6,102802806

	French Cubic Lines.	English Cubic Lines.
Centilitre	871,126926	1054,5643249

The *litre*, which is the unit for measures of capacity, = 2,1135 English pints, wine measure; and, therefore, 1 hectolitre equals 26,41906 wine gallons, or 2,83796 Winchester bushels. Hence 1 Winchester bushel = 35,236 litres.

SOLID OR CUBIC MEASURE.

	French Cubic Feet.	English Cubic Feet.
Decastere	291,738519	353,1700
Stere (metre cube)	29,1738519	35,3170
Decistere	2,91738519	3,5317

This measure is used for fire-wood, stone, &c. The *stere* is the same as the kilolitre in the preceding article.

WEIGHTS

	Poids de Marc.				English Troy Weight.			
	Livres.	oz.	gros	grains,	lbs.	oz.	dwt.	grs.
Bar (cubic metre of water)	2042	14	0	14	2679	6	3	8
Myriagramme	20	6	6	63,5	26	9	10	20
Kilogramme	2	0	5	35,15	2	8	3	2
Hectogramme		3	2	10,715		3	4	7,40
Decagramme			2	44,2715			6	10,34
Gramme				18,82715				15,434
Decigramme				1,882715				1,5434
Centigramme				0,1882715				0,15434

The above weights, as before stated, are deduced each from some decimal proportion of the metre. Thus the kilogramme corresponds with the contents of a cubic vessel of pure water at the lowest temperature, the side of which cube is the *tenth* part of a metre, (the decimetre,) and the gramme answers to the contents of a cubic vessel, the side of which is the *hundredth* part of the metre, (the centimetre;) for the contents of all cubic vessels are to each other in a triplicate ratio of their sides, (*Euclid*, 33, xi.)

ASTRONOMICAL MEASURES.

CENTESIMAL DIVISION OF TIME COMPARED WITH SEXAGESIMAL.

Centesimal.		Sexagesimal. H.	M.	S.
	1 second	0	0	0,864
100 seconds	1 minute	0	1	26,4
100 minutes	1 hour	2	24	0
10 hours	1 day	24	0	0

Sexagesimal.		Centesimal. H.	M.	S.
	1 second	0	0	1,1574
60 seconds	1 minute	0	69	54,4
60 minutes	1 hour	1	15	74
24 hours	1 day	10	0	0

CENTESIMAL DIVISION OF THE CIRCLE COMPARED WITH SEXAGESIMAL.

Centesimal.		Sexagesimal. °	′	″
100 seconds	1 minute (of space)	0	0	32⅖
100 minutes	1 degree	0	54	0
100 degrees	1 quadrant	90	0	0
400 degrees	1 circle	360	0	0

Sexagesimal.		Centesimal. °	′	″
60 tierces	1 second	0	0	8 7/81
60 seconds	1 minute of space	0	1	85 5/27
60 minutes	1 degree	1	11	11 1/9
90 degrees	1 quadrant	100	0	0
4 quadrants or 360°	1 circle	400	0	0

GEOGRAPHICAL AND ITINERARY MEASURES.

		English Miles.
Quarter of the terrestrial meridian	(10 millions of metres) =	6213,857
A degree	(1/90 part)	69,0429
Marine league	(1/20 of a degree)	3,4521
Marine mile	(minute, 1/60 of a degree)	1,1507
League of 25 to a degree		2,7617
A degree decimal	(1/100 of the quadrant)	62,13857
Myriametre	(1/10 of a degree decimal)	6,213857
Kilometre	(minute of a degree decimal)	0,62138
Mean league	(22½ to a degree)	3,10693
Post league	(2000 toises)	2,3000

TABLE OF MISCELLANIES.

The size of a printed book is determined and expressed by the folding of the paper on which it is printed; thus, if the sheet is folded into

2 leaves, or 4 pages, it is called	folio.
4 leaves, or 8 pages	quarto.
8 leaves, or 16 pages	octavo.
12 leaves, or 24 pages	duodecimo.
18 leaves, or 36 pages	eighteens, (decimo-octavo.)
24 leaves, or 48 pages	twenty-fours, (vigesimo-quarto.)
24 sheets of paper make	1 quire.
20 quires	1 ream.
40 quires, brown do.	1 bundle.
10 reams, or 200 quires	1 bale.
12 skins of parchment	1 dozen.
5 dozen, or 60 skins do.	1 roll.
12 things of one kind	1 dozen.
12 dozen, or 144 things	1 gross.
12 gross, or 144 dozen	1 great gross.
20 things of one kind	1 score.
5 score	1 hundred.
Load of bricks	500.
Load of tiles	1000.
Stone (a weight)	14 lbs.
Stone of meat or fish	8 lbs.
Peck of flour or salt	14 lbs.
Bushel of rock salt	65 lbs.
Bushel of crushed do.	56 lbs.
Gallon of train or seed-oil	7½ lbs.
Gallon of water	8 lbs. 5 oz. 6¼ dr.
Firkin of butter	56 lbs.
Barrel of do.	224 lbs.
Barrel of gunpowder	100 lbs.
Barrel of raisins.	112 lbs.
Barrel of flour	196 lbs,
Barrel of beef or pork	200 lbs.
Barrel of soap	256 lbs.
Fagot of steel	120 lbs.
Fother of lead	19½ cwt.
Square of flooring	10 feet square, or 100 square feet.

A palm is 3 inches, a span 9, and a cubit 18 inches.

A Scripture or Jewish cubit is nearly 22 inches.

A perch of stone, is a solid pile 16½ feet long, 1 foot high, and 1½ feet thick, containing 24¾ cubic feet.

BOOK V.

INVOLUTION—EVOLUTION—EXTRACTION OF THE SQUARE ROOT—EXTRACTION OF THE CUBE ROOT—EXTRACTION OF THE ROOTS OF ALL POWERS—OF AN EQUATION—GEOMETRICAL PROPORTION—COMPOUND PROPORTION—POSITION—ALLIGATION — PERMUTATION — ARRANGEMENT — COMBINATION—NEWTON'S BINOMIAL FORMULA—ARITHMETICAL PROPORTION AND PROGRESSION—GEOMETRICAL PROGRESSION—LOGARITHMS—USE OF THE TABLE.

SECTION XIX.

INVOLUTION—EVOLUTION—EXTRACTION OF THE SQUARE ROOT.

Involution, or Raising of Powers.

354. This consists in *the continual multiplication of a quantity into itself.* The successive products are called *the powers of the given quantity,* and each shows, by its appellation, the number of times that it contains the given quantity as factor, the quantity itself being the first power or root. Thus, let a be any quantity; then, a is the first power or root of all the powers of a; $a \times a$, or aa, is the second power or square of a; $a \times a \times a$, or aaa, is the third power or cube of a; $a \times a \times a \times a$, or $aaaa$, is the fourth power of a. This being the product of two squares, is also called the *biquadrate,* from *bis* and *quadrans.* Other names have been given to several of the higher powers, indicative of their formation from the square and cube; but these have generally fallen into disuse, and $a \times a \times a \times a \times a$, or $aaaaa$, is simply called the fifth power of a; $aaaaaa$, the sixth, and so on. For the sake of convenience, these products are written successively a, a^2, a^3, a^4, a^5, a^6, &c., and are read, a, a squared, a cubed, a fourth, a fifth, a sixth, &c. The figure on the right is called the *index*

or *exponent,* because it shows the number of times that a is factor in each product. Hence, it is evident, that if we add the indices of two or more powers of a quantity, their sum will be *the index of the power to which the quantity would be raised by multiplying those powers together.* The scholar must remember, that the exponent of the root, or quantity itself, though seldom written, is *a unit,* and must be so considered in adding or subtracting the exponents. Thus, $a \times a^2$, or $aa^2 = a^3$; $a^2a^2 = a^4$; $a^2a^3 = a^5$; aa^2a^3, $a^2a^2a^2$ or $a^3a^3 = a^6$; $a^2a^2a^2a^2$, aa^2a^5, $a^2a^2a^4$, $a^2a^3a^3$ or $a^4a^4 = a^8$.

355. From the above it is plain, that a quantity may be raised to the higher powers *in various ways,* and the higher the power, *the greater the variety.* Thus, to find the twelfth power of 2, as we know that the cube of 2 is 8, we say $8 \times 8 \times 8 \times 8 = 4096$, which is easier than to perform the continual multiplication $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 4096$.

Because either of two numbers taken from their sum leaves the other, the index of either of two factors taken from the index of their product will leave *the index of the other factor.* But the product divided by either factor gives the other; therefore, to divide any higher power of a quantity by a lower, we have only to *subtract the index of the divisor from the index of the dividend, and the remainder will be the index of the quotient.* Thus, $\frac{a^2}{a} = a$; $\frac{a^3}{a} = a^2$; $\frac{a^5}{a^2} = a^3$; $\frac{a^5}{a^3} = a^2$; also, $\frac{2^7}{2^5} = 2^2 = 4$; $\frac{3^9}{3^6} = 3^3 = 27$; $\frac{5^{11}}{5^7} = 5^4 = 625$; $\frac{a^2a^3a^4}{a\,a^2a^3} = a^3$.

356. A vulgar fraction is raised to any power *by raising each of its terms to that power.* For example, the fifth power of $\frac{2}{3}$ may be expressed thus: $\left(\frac{2}{3}\right)^5$ or $\frac{2^5}{3^5}$, and is equal to $\frac{32}{243}$. This is obvious from the rule for multiplying fractions.

A mixed number may be reduced to an improper fraction, and then involved as a fraction. Thus, $(2\frac{1}{2})^5 = (\frac{5}{2})^5 = \frac{5^5}{2^5} = \frac{3125}{32} = 97\frac{21}{32}$.

357. A decimal fraction, whether finite or infinite, is involved by the rules already given for the multiplication of decimals. We may, however, remark, that a finite decimal may be involved as a whole number, after which, the number of

decimal places, of which the result must consist, is determined *by multiplying the number of decimals in the given number by the index of the power to which it is raised.* Thus,

$$(,04)^3 = ,000064$$

Having found that the cube of 4 is 64, as the given number contains 2 decimals, we multiply 2 by 3, the index, and have 6 for the number of decimal places in the result. Then, as 64 contains only two figures, we subtract 2 from 6, and the remainder 4 is the number of ciphers which we must prefix to 64. The cube of ,04 is therefore 64 millionths, as above.

The reason of the above operation will be seen by reflecting, that as the factors are alike, and as the product must contain as many as are contained in both, the square will contain twice as many as the given number, the cube, 3 times as many, &c.

The scholar may verify the following results:

$$3^6 = 729;\ 2^{15} = 32768;\ 5^7 = 78125$$
$$\frac{5^5}{7^5} = \frac{3125}{16807};\ (\tfrac{7}{11})^8 = \frac{5764801}{214358881}$$
$$(,4)^3 = ,064;\ (,5)^6 = ,015625$$
$$(,05)^5 = ,0000003125$$
$$(3\tfrac{3}{8})^3 = 38\tfrac{227}{512};\ (,\dot{6}\dot{3})^4 = ,16399 +$$
$$\frac{7^9}{7^5} = 2401;\ (\tfrac{5}{11})^5 \div (\tfrac{5}{11})^2 = \tfrac{125}{1331}$$
$$(2\tfrac{1}{2})^7 \div (2\tfrac{1}{2})^3 = 39\tfrac{1}{16}$$

Powers of the Nine Digits, to the Tenth Power, inclusive.

Roots.	1	2	3	4	5	6	7	8	9
2d. Power.	1	4	9	16	25	36	49	64	81
3d.	1	8	27	64	125	216	343	512	729
4th.	1	16	81	256	625	1296	2401	4096	6561
5th.	1	32	243	1024	3125	7776	16807	32768	59049
6th.	1	64	729	4096	15625	46656	117649	262144	531441
7th.	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th.	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th.	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th.	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401

Evolution, or Extraction of Roots.

358. Evolution is *the method of finding a quantity from any given power of that quantity.* The quantity so found is called the root of the power from which it is extracted, and assumes its name accordingly: that is, if found from the square, it is called *the square root,* if from the cube, *the cube root,* &c.

Extraction of the Square Root.

359. If the square of a number consists of only two figures, *its root cannot consist of more than one,* and must, in whole numbers, be one of the nine digits. This is evident, because the product of two factors (121) contains as many figures as both of them or one figure less; and therefore, if the root or quantity contained two figures, the product, or square, would contain at least three. Again, if the square contains three figures, *the root must contain at least two,* because (121) the product cannot contain more figures than both factors, which would be the case if the root contained only one figure.

It is evident, therefore, that if the square contains four figures, *the root will contain two*; if five figures, *the root will contain three,* &c.; and, consequently, that the number of figures of which the root will consist, may be determined *by separating the figures of the square into periods of two figures each.*

360. When the root contains more than one figure and has its unit figure significant, it may always be considered as *a number of tens and units.* Now a quantity, consisting of two parts, is called *a binomial,* from *bis* and *nomen,* a name, and the square of a binomial contains the square of each of its terms, together with twice their product. Let the root consist of tens and units, and let a represent the tens, and b, the units; then, to find the square of the binomial $a + b$, we involve it thus, (See art. 138:)

$$\begin{array}{lll} a + b & & \\ \underline{a + b} & & \\ a^2 + ab & & \\ \quad\;\; \underline{ab + b^2} & & \\ a^2 + 2ab + b^2 & & \end{array}$$

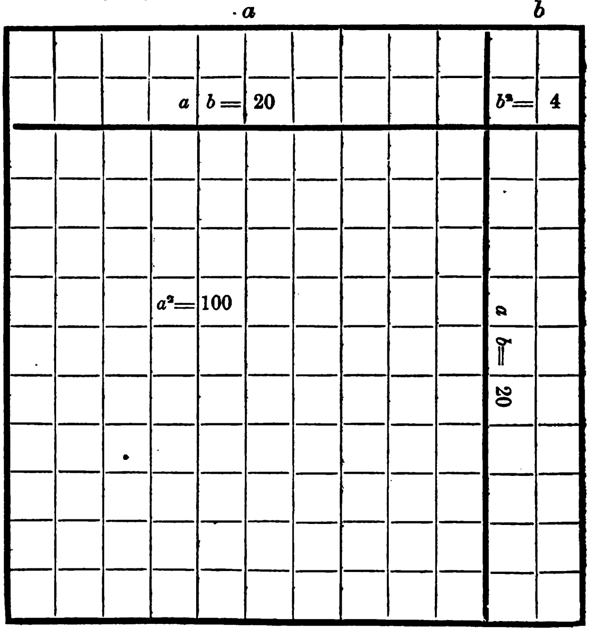
The binomial $a + b$, being multiplied first by a, and then by b, we have for the sum of the products $a^2 + 2ab + b^2$; that is to say, *the square of the tens, the square of the units, and* *twice the product of the tens and units*, and this is the formula by means of which the square root is extracted.

This may be exemplified by figures, thus: in squaring 12, the number of inches in a foot, we have 144, the number of square inches in a square foot: but 12 consists of 10 + 2, and in squaring this binomial, as above, we have

$$\begin{array}{r} 10 + 2 \\ 10 + 2 \\ \hline 100 + 20 \\ 20 + 4 \\ \hline 100 + (20)\,2 + 4 \end{array}$$

or $100 + 40 + 4 = 144$; that is to say, the square of the tens, twice the product of the tens and units, and the square of the units.

Or we may represent the square foot geometrically, thus:



In which the same parts are clearly developed, their sum being 144, as before.

361. To extract the square root, therefore, of any number, the root being binomial, we reason thus: Of whatever number of figures the root may consist, the square of its tens—that is, the square of the number on the left of its unit figure—must be a number of hundreds, because $10 \times 10 = 100$, and therefore cannot form any part of the two right-hand figures of the square. Again, if the part of the root, considered as tens, contains more than one figure, the part on the left of its right-hand figure is a number of tens, with regard to that figure, or a number of hundreds, with regard to the first: therefore, as $100 \times 100 = 10000$, the square of this part, being a number of tens of thousands, cannot form any part of the first four figures of the square. Continuing to reason thus, it is evident that, in order to find that part of the square which contains the square of the left-hand figure of the root, *we point off the figures of the square into periods of two figures each, beginning at the right hand.* This, therefore, is the first step.

As the square of the tens is a number of hundreds, and as the square of the units (121) cannot exceed two figures, it is evident that the square of the units can form no part of the square of the tens.

Again, the addition of $2\,ab$ to the square of the tens, cannot increase the root of the tens, by a single unit, because this unit would be 10, and the difference between a^2 and $(a+10)^2$ is $20a+100$; which, as b is less than 10, is greater $2ab$. Therefore, having pointed off the given number, to find the first figure, or tens of the root, *we extract the root of the greatest square contained in the left-hand period of the square,* and place it as we do a quotient figure in long division. This is the second step.

We square the root found, and subtract its square from the left-hand period of the given number. To the right of the remainder we bring down the next period of the square. This is the third step.

The number last formed is called *the new dividend.*

Having subtracted the square of the tens, or a^2, it is evident that the remaining part of the square contains $2ab + b^2$. It is also evident that, as $2ab$ is a number of tens, it cannot form part of the right-hand figure of the new dividend, for which reason *we point off that figure,* and consider the part on the left *as containing* $2a \times b$. Now the figure already found in the root is a; we therefore double it, and place the product, as a divisor, on the left of the new dividend. Then, as the product of two numbers, divided by one of them, gives the other, we divide the left-hand part of the dividend by the double of the root found, which is $2a$, and the quotient, or unit figure b, we place in the root, on the right of a, and also on the right of the divisor $2a$. This may be called the fourth step.

Lastly, we multiply the divisor by the quotient figure b, and it is easy to see that as this figure is placed on the right in the divisor, we have in the product *the square of* b *and* $2ab$. This, therefore, completes the square. We then subtract, and if the given number is a perfect square consisting of two periods there will be no remainder, and the work is done. This may, therefore, be called the last step.

If in multiplying the divisor by b, the product is greater than the new dividend, *we diminish* b *till the subtraction becomes possible.*

Examples.

Required the square root of 225 :

$$\begin{array}{rl}
 & a+b \\
 & 2,25\ (15 \text{ square root.} \\
a^2 = & 1 \\ \cline{2-2}
2a+b=25 \quad & 12,5 \text{ new dividend.} \\
 & 12\ 5 = 2ab + b^2 \\ \cline{2-2}
 & *\!*\,*
\end{array}$$

Proof: $15 \times 15 = 225$.

In this example, the first step is to point off the two right-hand figures 25 of the square; the second, to find the greatest square in 2, which constitutes the left-hand period, and as this is 1, or rather 1^2, (169,) we place its root 1 in the quotient, as in division. The third step is to subtract the square of 1, or a^2, from 2, the left-hand period, and to bring down the period 25 on the right of the remainder, which makes the new dividend 125. The fourth step is to point off the right-hand figure 5 of this dividend, and divide 12, the part on the left, by 2 tens, or $2a$, the double of the root found. Now, though the divisor 2 is contained 6 times in 12; as we must place 6 on the right of the divisor 2 tens, in order to have the square of the units, it is easy to see that because 6 is not contained 6 times in 5, the figure pointed off, the product would be too great. We therefore place only 5 in the root, and on the right of the divisor 2 tens. The last step is to multiply the divisor

25 by 5, the unit figure of the root, which gives 125; that is to say, $2ab + b^2$. We then subtract, and as there is no remainder, we find that 225 is a perfect square, and the square of 15. To prove the work, we square the root 15, which gives the square 225.

362. From what has been said, we easily perceive that, if the square consists of more than two periods, the root will contain more than two figures. In this case, we find the two first figures of the root, as in the preceding article. Then, if we seek another figure in the root, it is evident that the two figures already found are tens with regard to that figure. It is also evident that the square of the part of the root already found, which, in accordance with the general formula, we may call a^2, has already been subtracted from the given square: seeing that we have subtracted the square of its tens, twice the tens multiplied by the units, and the square of the units. Therefore, in seeking a new figure of the root, we bring down the next period to the right of the remainder, which forms a new dividend; and as this dividend, from what has been said above, contains $2ab + b^2$,—that is to say, the double of the two figures, or tens, already found, multiplied by the unit figure which we are seeking, and the square of that unit figure, it is plain that, to find the new figure, we must point off the unit figure of the new dividend; and, taking the double of the two figures, or root, already found, for a new divisor, the new figure will be found in dividing the left-hand part of the new dividend by this divisor, and the operation is completed as before.

Having thus taken away the square of the tens of the root, which now consists of three figures, and may be called a, together with the square of the units, and twice the product of the tens and units, we have taken away the square of this root; and it is easy to see that, in seeking still another figure of the root, the part on the left—that is, the three figures now found—will be a number of tens with regard to it; and as the square of these three, or a^2, has already been subtracted, *we form a new dividend, double the root already found, for a divisor,* and proceed to complete the operation, as usual. Wherefore, in seeking a new figure of the root, the double of the root already found, or $2a$, is the general divisor; and to form this, we have only to *double the right-hand figure of the preceding divisor,* carrying a unit, when necessary, to the next figure on the left. In this manner, therefore, we can extract the root of a square consisting of any number of periods.

Example.

Required the square root of 81632489796:

	8,16,32,48,97,96 (285714
	4
48	41,6
	38 4
565	323,2
	282 5
5707	4074,8
	3994 9
57141	7999,7
	5714 1
571424	228569,6
	228569 6
	*******

363. The square being formed by the multiplication of a quantity into itself, the root of a square is *one side of the figure, or it is the number which, multiplied into itself, will produce the given square.* A prime number, therefore, not being divisible by any other number, cannot be the square of any whole number, and therefore cannot have an integral root. For example, we cannot, in whole numbers, have the square root of 2; because no whole number multiplied by itself will produce 2. Again, as the square of 1 is 1, it is plain that the square root of 2, if possible, must be a quantity between 1 and 2; it is, therefore, 1 and a fraction. Suppose this mixed number reduced to an improper fraction and expressed in its lowest terms: then these terms are prime to each other, and the square of this fraction; that is to say, the square of the numerator divided by the square of the denominator must be equal to 2. But we have seen (179) that the squares, cubes, &c. of numbers, prime to each other, are still prime to each other; and, therefore, it is impossible that they can have any common factor, much less is it possible that one can be a multiple of another. From this it is evident that the square root of 2 cannot be expressed by any definite number, whole, fractional, or mixed.

In the same manner it may be shown that *the square root of any prime number whatever is impossible.* These roots are therefore called *surd*, irrational, or incommensurable.*†

364. We have seen that the construction of decimals is the same as that of whole numbers; hence, a finite decimal number, whether fractional or mixed, may be read as a number of units of its last decimal order. The part on the left of the last decimal order is therefore a number of tens with regard to that order. Hence also it is plain that, if we fix upon any figure in a decimal number, the whole of the figures on the left, including the integers, constitute a number of tens with regard to that figure; consequently the square root of a decimal number is extracted, by means of the formula $a^2 + 2ab + b^2$, precisely *in the same manner as that of a whole number.*

365. As we point off a whole number, beginning at the right, it is plain that, having extracted the root of the integral part of a decimal number in the usual manner, we must, to form a new dividend, bring down a new period; that is to say, two decimal figures; and, as these (359) will give but one figure in the root, we must, for every decimal figure required in the root, have *two decimal figures in the square.* This is evident also from the multiplication of decimals, (234.) To extract the square root of a decimal number, therefore, we begin at the comma and point off the integral part, as usual, towards the left; then, recommencing at the comma, we point off the decimal part into periods of two figures each, proceeding towards the right, and making as many periods on the right of the comma (234) as we would have decimal figures in the root. This we can easily do *by placing ciphers on the right of the decimal part.*

366. From the above we easily infer that, though we cannot definitely express the value of the mixed number which is the square root of 2, or that of any other surd, *we can approach to it, decimally, as near as we please;* that is to say, so that the error in the square may be less than any assignable quantity.

This we shall endeavour to elucidate. Suppose the root, already found, to be integral, its last figure being units, and call this part of the root a. The part wanting to complete the root, or b, is (220) less than a unit. Suppose it a unit;

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* Absurd. † Not to be measured.

then, as $2ab$ is (361) the only part of the square which can affect the square of the tens—that is, of the root found; and, as b is a unit, $2ab$ is the same as $2a$: and therefore, as this is too great, the difference between the square of the root found and the true square, can never equal the double of the root found; even when the unit b is integral. But we are now speaking of b as fractional; and as a number multiplied by a fraction is (335) always diminished, it is plain that if $b = \frac{1}{10}$—that is, if we extract the root to one place of decimals—the part lacking in the square must be *less than* $\frac{1}{10}$ *of twice the root found*, or than $\frac{1}{5}$ of the root found. If we extract it to two places—that is, if $b = \frac{1}{100}$—the part lacking will be *less than* $\frac{1}{100}$ *of twice the root found*, or than $\frac{1}{50}$ of the root found. In the same manner, if we have 3, 4, 5, 6, &c. decimals in the root, the part lacking in the square will be *within* $\frac{1}{500}$, $\frac{1}{5000}$, $\frac{1}{50000}$, $\frac{1}{500000}$, &c. *of the root found;* and hence, by the number of units of which the integral root will consist, *we can judge beforehand of the error in the square, and extract the root accordingly to any proposed degree of exactness.*

Examples.

1. Required the square root of 2, so that the square of the root may be within $\frac{1}{1000}$ of the number 2.

By the above reasoning, if we extract the root to 4 decimals, its square will differ from 2 by less than $\frac{1}{5000}$ of its integral root; much more, then, will be within $\frac{1}{5000}$, and consequently within $\frac{1}{1000}$ of the number 2. Wherefore, placing 8 ciphers on the right of the 2, as decimals, or, which is better, conceiving them to be so placed, we proceed thus:

$$\begin{array}{ll}
 & 2\ (1{,}4142 \\
 & 1 \\ \cline{2-2}
24 & 10{,}0 \\
 & \ \ 9\ 6 \\ \cline{2-2}
281 & \quad 40{,}0 \\
 & \quad 28\ 1 \\ \cline{2-2}
2824 & \quad 1190{,}0 \\
 & \quad 1129\ 6 \\ \cline{2-2}
28282 & \qquad 6040{,}0 \\
 & \qquad 5656\ 4 \\ \cline{2-2}
 & \qquad 383\ 6
\end{array}
\qquad\qquad
\begin{array}{rr}
\text{Proof:} & 1{,}4142 \\
 & 1{,}4142 \\ \cline{2-2}
 & 28284 \\
 & 197988\ \ \\
 & 197988\ \ \ \ \\ \cline{2-2}
 & 1{,}99996164
\end{array}$$

The square of 1,4142 is, therefore, not only within $\frac{1}{1000}$ of the number 2, as required, but differs from 2, by less than $\frac{1}{10000}$ of a unit.

9. Required the square root of 143, such that the difference between the square of the root and 143 shall be less than $\frac{1}{1000}$ of a unit.

We first consider whether four places of decimals will be sufficient. To ascertain this, as we know that the root will be nearly 12, and that the square of the root extracted to 4 places will be within $\frac{1}{5000}$ of the integral root, we divide 11 by 5000, thus: $\frac{11}{5000}$, which fraction, being greater than $\frac{1}{1000}$, shows that the error in the square may be too great. Then, as 5 places will, by the above reasoning, give the error within $\frac{11}{50000}$, which is much less than $\frac{1}{1000}$ of a unit, we are sure that the root extracted to 5 places will be as near as we require; that is to say, the difference between its square and 143 will be less than $\frac{1}{1000}$ of a unit.

We therefore extract the root to 5 places of decimals, as follows:

	143,(11,95826
	121
229	220,0
	206 1
2385	1390,0
	1192 5
23908	19750,0
	19126 4
239162	62360,0
	47832 4
2391646	1452760,0
	1434987 6
	17772 4

Knowing that the square root of 143 must, in whole numbers, be 11, the square of 12 being 144, we place 11 in the root, at once, with a comma on the right; and, having subtracted its square, we double it for a divisor, and proceed as usual.

To verify the above method of ascertaining the approximation of the square of the root to the true square, or proposed number, we will now square the root, supposing it to contain four decimals, thus:

$$\begin{array}{r}
11{,}9582 \\
11{,}9582 \\
\hline
239164 \\
956656 \\
597910 \\
14230258 \\
\hline
142{,}99854724 \\
143{,} \\
\hline
{,}00145276
\end{array}$$

Whence we perceive that the error is too great, as was predicted.

Let us now square 11,95826: to do which we may consider the part 11,9582, the square of which we have already found, as tens, or a, and the last figure 6, as units, or b. The square of this binomial then contains $a^2 + 2ab + b^2$. Now $11{,}9582 = \frac{119582}{10000}$, and the figure $6 = \frac{6}{100000}$; therefore

$$\begin{array}{rr}
2ab = \dfrac{119582 \times 12}{1000000000} = & {,}001434984 \\
b^2 = \frac{6}{100000} \times \frac{6}{100000} = & {,}0000000036 \\
\text{and the square found above, } a^2 = & 142{,}99854724 \\
\hline
& 142{,}9999822276
\end{array}$$

Having added the several parts, the sum, which is the square of 11,95826, is not only true to within a thousandth, as was predicted, but even to less than a ten-thousandth. The scholar may verify the square, thus formed, by squaring 11,95826 in the usual way.

367. The periods of the square show not only the number of figures in the root, (361) but also the order of those figures. For, as the square of the left-hand figure is contained in the left-hand period, whatever may be the number of periods or figures, the square of any figure is contained in the period which answers to its ordinal place; that is, when a period is brought down to form a new dividend, the order of the figure resulting from its division, is found *by beginning at the right hand and counting on the periods of the square, as if they were single figures, units, tens, hundreds, &c., till we rest on the period brought*

down; the name pronounced on this is the order of the figure resulting from its division. Consequently, when any new dividend, after pointing off its right-hand figure, does not contain the double of the root, or general divisor, *we must place a cipher in the root and on the right of the divisor, bring down another period, and proceed as usual.*

368. The square of tenths being hundredths, (234,) the square of hundredths, ten-thousandths, &c., it is plain, that the order of the decimal figure, resulting from the division of any decimal period, may be determined *by beginning at the comma and counting on the periods, as though they were single figures, till we rest on the period in question, tenths, hundredths, thousandths, &c.;* and therefore, that the above rule, for placing a cipher in the root, applies also to decimals. This is analagous to the rule for placing a cipher in the quotient in division. (151.)

For example, if, in extracting the square root of 143, as in the preceding article, we seek another decimal figure, as the new divisor is 2391652, and the new dividend, after cutting off a cipher on the right, is 1777240, we find, that this decimal figure must be a cipher; we should therefore, to continue the operation, place a cipher in the root and on the right of the divisor, and another period, that is, two ciphers, on the right of the new dividend. The new divisor would then be 23916520, and the new dividend 1777240000, with which we should proceed as usual.

369. Having obtained several figures of the root, we can, simply by division, find as many more, minus one, as follows: Form the new divisor, as usual, place one cipher on the right of the remainder, instead of two, for a new dividend; then if the divisor is not contained in it, place a cipher in the quotient, or root, as we should do in the ordinary method; cut off one figure on the right of the divisor, and divide by the divisor thus diminished. Should the divisor be still too great, place another cipher in the root, and cut off another figure, and so on, till the divisor is contained. Then proceed in the division, as in articles 244 and 245, cutting off a figure of the divisor, at each partial division, and taking notice in the multiplication, of what would be to carry from the figure last cut off. Thus continue the operation *till the quotient contains as many figures, minus one, as were previously found in the root;* or, which is the same thing, till the divisor consists *of only two figures, or three figures, if the double of the first figure of the root exceeds*

10. The figures of the quotient, including the ciphers, if any, are so many new figures to place in the root, on the right of those previously found.

370. The student will perceive, that by this method, we do not subtract the square of the part of the root which we are seeking. But we have seen (361) that this cannot affect the square of the tens; therefore, it cannot affect the dividend which we are dividing. Again, in the division, as we place a figure in the quotient before we cut off a figure from the divisor, and, subsequently, a figure, for every figure cut off, it is plain, that in limiting ourselves to one figure less than the root already found, we shall, at the end of the operation, have, at least, two figures of the divisor left. Now, as the operation could (247) be carried one place farther and give the quotient correct within a unit of the next decimal order, this method will always give a true result.

For example, having found the first seven figures 11,95826 of the root of 143, as in art. 366, we find the next six figures of the root thus:

$$\begin{array}{r l}
23,9,1,6,5,2) & 1777240\,(074310 \\
 & 103084 \\
 & \;\;7418 \\
 & \;\;\;243 \\
 & \;\;\;\;\;4
\end{array}$$

The new divisor not being contained in the new dividend, we place a cipher in the quotient. We then cut off one figure from the right of the divisor, and, as the divisor thus reduced is contained 7 times, we place 7 in the quotient. Then, as the figure 2, which is cut off, multiplied by 7, gives 1 to carry, we say, 7 times 5 is 35, and 1 is 36; 36 from 40 leaves 4, which we write underneath and carry 4, as is usual in this mode of contracting a division. We thus, in multiplying each new divisor by its quotient figure, *add what is to carry from that figure of the divisor which was last cut off.* In other respects we proceed as usual. Having obtained the six figures 074310, we place them on the right of the root found, which becomes 11,95826074310. As we now have 11 decimals, its square (366) will be within $\frac{1}{500000000000}$ of the integral root; hence, it will differ from 143 by less than $\frac{1}{10000000000}$ of a unit.

To have more figures in the root, by the above method, we square 11,95826074310, and find that the square, which is 142,9999999999665641976100, is not only within $\frac{1}{10000000000}$

of the number 143, as was predicted, but even within $\frac{1}{10000000000}$.

We then find the difference between this square and 143. Now this is evidently the same as the difference between the number on the right of the figure 9 which is most remote from the comma, and a unit of the order of this 9. This difference, that is to say, the difference between any number and its alt, is the arithmetical complement of that number; we therefore write the number 665641976100 and find for its arith. comp. 334358023900. Now a cipher placed on the right of this, forms the new dividend, and the double of the root or new divisor is 2391652148620. But as the equimultiples of two numbers contain each other as often as the numbers do, we omit the cipher on the right of the divisor, and also the cipher which we should place on the right of the complement, which, therefore, we take for a new dividend, and proceed, as in the previous division, thus:

```
23,9,1,6,5,2,1,4,8,6,2) 334358023900 (13980211298
                         95192809038
                         23443244580
                          1918375243
                             5053525
                              270221
                               31056
                                7140
                                2357
                                 206
                                  15
```

and placing the quotient, as before, on the right of the root, we have 11,9582607431013980211298 for the root; the square of which, as we now have 22 decimals, will (366) differ from 143, by less than *one hundred quintillionth* of a unit. The square in effect is 142,999999999999999999999025253571839367268448O4; which, as we have 21 nines, differs from 143 by less than *a sextillionth* of a unit, and is nearer than was predicted.

371. At the next operation, we could easily find 23 new figures of the root, which would then contain 47 figures, and its square (366) would consequently differ from 143 by less than *a ten-tredecillionth of a unit;* and, consequently, the number 142 would be followed by at least 43 nines. Thus, though the root

can be found within any prescribed limit, *it never can be exactly ascertained.*

The student will, however, have some idea of the minuteness of the unit which would then be wanting to complete the square, when he is informed that, if we suppose the whole globe we inhabit to be composed of sand, such that a cubic inch would contain ten millions of grains, a ten-tredecillionth of the globe would be less than the billionth part of one of these extremely minute grains, which part, could the grain be so comminuted, could never be discovered, even by the aid of the most powerful microscope.

372. If we square a fraction, the numerator of the square is the square of the numerator of the fraction, and the denominator of the square, the square of the denominator of the fraction. The square of a fraction, therefore, is *the square root of its numerator, divided by the square root of its denominator;* consequently, to find the root, when the numerator and denominator are squares, we extract the root of each, and the work is done. But, as this is seldom the case, for a general rule, we may *find its equivalent decimal fraction, and extract the root of the latter,* taking care to have twice as many decimal places on the right of the comma, in the equivalent, or square, as we would have decimals in the root. Also, from what has been said in the preceding articles, it is plain, that if the first period on the right of the comma consists of ciphers, the first figure of the root, that is, the figure in the place of tenths, will be a cipher, and that, if we have several such periods in succession, on the right of the comma, we must *place one cipher in the root for each,* that is to say, between the comma and the first significant figure; and, if on the periods of the square, we count, as though they were single figures, tenths, hundredths, thousandths, &c., the first significant figure of the root will be *of the order pronounced on the period from which it is derived.*

373. The following rule is also general: *Multiply the numerator of the given fraction by its denominator, and divide the square root of the product by the denominator.* This last rule is thus explained: if we multiply both terms by the denominator, we do not alter the value; therefore, the square root of the new fraction is the square root of the given one. But the square root of the new fraction is the square root of its numerator (which is the product mentioned in the rule) divided by the square root of the denominator; that is, by the denominator of the given fraction.

To extract the square root of $\frac{9}{25}$, we take the square root of 9 for a numerator, and the square root of 25, for a denominator. The square root of $\frac{9}{25}$ is therefore $\frac{3}{5}$. Proof: $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$.

But, if we require the square root of $\frac{3}{7}$, as neither term is a square, we multiply 3 by 7, and, if we want the root so near that its square may differ from $\frac{3}{7}$ by less than a hundred-thousandth of a unit, we extract the root of 21 to 6 places of decimals, and divide it by 7. For, though the root of $\frac{3}{7}$ must (207) be greater than $\frac{3}{7}$, it cannot equal a unit, because its square would then be a unit, or more; therefore, the square of the root, being within $\frac{1}{5}$ of the root found, that is, within $\frac{1}{5}$ of a unit, for one decimal, $\frac{1}{50}$ for two, &c., (366,) will, if extracted one place farther than the proposed limit, always be within that limit, and it is evidently the same for any proper fraction.

Operation.

```
              21 (4,582575 + square root of 21
85             50,0              7)4,582575 +
908             750,0              ---------
9162             2360,0             ,654653 + square root of 3/7.
91,6,4           ) 52760 (575
                    6940
                     526
                      68
```

The work is here contracted by omitting to write the products, as well in the extraction as in the division: the subtraction being in both cases performed mentally.

Or, we proceed thus:

$\frac{3}{7} = ,\dot{4}2857\dot{1}$ (318) and extracting the root as follows:

```
                ,4̇28571̇ (,654653 +
    125           68,5
    1304           607,1
    13086           8554,2
    1309,2)          70268 (53
                      4808
                       881
```

We have the same result as before.

The only difference between the extraction of the root of this, and the extraction of that of an ordinary decimal is, that in extending the operation, instead of annexing ciphers, *we repeat in their proper order, the figures of the period.*

To verify the approximation, we square the root thus:

$$\begin{array}{rl}
,654653 & \\
\underline{,654653} & \\
1963959 & \\
3273265 & \\
3927918 & \\
35351262 & \quad 54 \text{ times.} \\
\underline{3927918} & \\
,428570550409 &
\end{array}$$

The difference between this square and $\frac{3}{7}$ is ,000000878162+, which plainly shows that it is within the prescribed limit.

374. The sign $\sqrt{}$, called *the radical sign*, from the Latin *radix*, a root, is placed on the left of a quantity, to denote the extraction of some root of that quantity. Alone, it signifies the square root; and any other root is designated by placing over the sign an index corresponding to the index of the power, the root of which is to be extracted. Thus, for the square root, we write $\sqrt{}$, or $\sqrt[2]{}$; for the cube root, $\sqrt[3]{}$; for the fourth root, $\sqrt[4]{}$, and so on. Hence, $\sqrt{\frac{64}{121}}=\frac{8}{11}$; $\sqrt[2]{\frac{169}{196}}=\frac{13}{14}$; $\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$.

Examples.

1. $\sqrt{2209}=47$
2. $\sqrt{4096}=64$
3. $\sqrt{32761}=181$
4. $\sqrt{51529}=227$
5. $\sqrt{5764801}=2401$
6. $\sqrt{328115878596}=572814$

7. Required the square root of 5, so near, that its square may differ from 5 by less than a ten-thousandth.

Ans. 2,23606.

8. Required the square root of 13, so near, that its square may differ from 13 by less than a hundred-thousandth.

Ans. 3,605551.

9. Required the square root of 3, so near, that its square may differ from 3 by less than a millionth.

Ans. 1,7320508 +

10. Required the square root of 19, so near, that its square may differ from 19 by less than a quadrillionth.

Ans. 4,3588989435406735; the square of which differs from 19 by 45539145116639775 hundred-nonillionths, or rather less than half a quadrillionth.

Required the square root of the following fractions, so near, that the square of each root may differ from the fraction which gave it by less than a millionth.

11. $\sqrt{\frac{5}{6}} = {,}9128709$
12. $\sqrt{\frac{2}{3}} = {,}8164965$
13. $\sqrt{\frac{7}{9}} = {,}8819171$
14. $\sqrt{\frac{8}{11}} = {,}8528028$
15. $\sqrt{\frac{1}{13}} = {,}2773500$
16. $\sqrt{\frac{1}{999}} = {,}0316385$

17. Having ten square lots of ground, the first containing one acre, the second, two, the third, three, &c., to the tenth inclusive; required the length, in ft. and inches, of one side of each.

Ans. No. 1. 208 ft. 8½ in. +
No. 2. 295 ft. 2 in. —
No. 3. 361 ft. 5½ in. +
No. 4. 417 ft. 5 in. +
No. 5. 466 ft. 8 in. +
No. 6. 511 ft. 3 in. —
No. 7. 552 ft. 2 in. +
No. 8. 590 ft. 4 in. —
No. 9. 626 ft. 1½ in +
No. 10. 660 ft.

18. Having 6 acres of ground which we wish to plant with apple-trees, in equidistant rows, at right angles to each other, at what distance must the trees be placed, so that we may have 20 on the first acre, 30 on the second, 40 on the third, 50 on the fourth, 60 on the fifth, and 70 on the sixth?

Ans. On the 1st acre at 46 ft. 8 in. +
" 2d " 38 ft. 1 in. +
" 3d " 33 ft.
4th at 29 ft. 6 in. +
5th " 26 ft. 11 in. +
6th " 24 ft. 11 in. +

19. Wishing to plant 529 trees in equidistant rows, at right angles to each other, so that the orchard may form a square, and have 35 trees to the acre; we would know the number of rows, the number of trees in each row, the distance, in feet and inches, from tree to tree, and how much land the orchard will contain.

Ans. 23 rows; 23 trees in each; 35 ft. 3 in. + apart, and 15 A. 18 P. + in the orchard.

SECTION XX.

EXTRACTION OF THE CUBE ROOT—EXTRACTION OF THE ROOTS OF ALL POWERS.

Extraction of the Cube Root.

375. The cube of a number being a product in which the number is three times factor, the cube root of a number is *the number, which being written three times and involved, produces the given number.*

Now the cube of one figure can never consist of more than three figures, (121,) neither can the cube root of a number expressed by three figures consist of more than one figure; for, if we suppose it to consist of two, its square (121) must contain at least three, and its cube four or five. The cube root, therefore, of a number, which is a perfect cube, and which does not contain more than three figures, must be one of the nine digits, and may easily be found by the table of powers. (357.)

376. If the cube consists of more than three figures, the root will contain *two or more*, because the product of two numbers cannot contain more figures than both factors. Now, if the root consists of two or more figures, we may consider it as a binomial, that is, as consisting of tens and units; then, calling the tens a, and the units b, we shall cube $a + b$ by multiplying its square, which is already known, by $a + b$, thus:

$$\begin{array}{l}
a^2 + 2ab + b^2 \\
a + b \\
\hline
a^3 + 2a^2b + ab^2 \\
\qquad\; a^2b + 2ab^2 + b^3 \\
\hline
a^3 + 3a^2b + 3ab^2 + b^3
\end{array}$$

from which we find, that the cube of a number, consisting of tens and units, contains *the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units.* By this formula, which shows the constituent parts of the cube, the cube root is extracted.

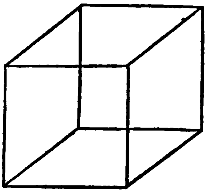
377. We may also verify this formula arithmetically, by applying its several parts to the dimensions of a foot; namely, (10 + 2) inches, thus:

$$\left.\begin{array}{rcl}
a^3 = 10^3 & = & 1000 \\
3a^2b = 3 \times 10^2 \times 2 & = & 600 \\
3ab^2 = 3 \times 10 \times 2^2 & = & 120 \\
b^3 = 2^3 & = & 8 \\
\hline
\end{array}\right\} \text{cubic inches.}$$

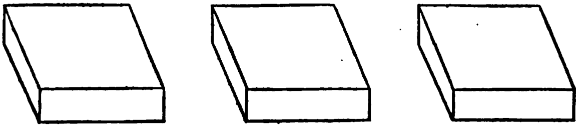
The sum of the several parts being 1728 cubic inches, the true content of a cubic foot, shows the correctness of the formula.

378. We may also verify the formula geometrically by constructing a cube according to its several parts, thus:

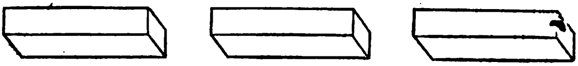
Let a^3 be represented by the following cube:



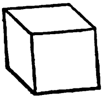
$3a^2b$, by the three following parallelepipeds:



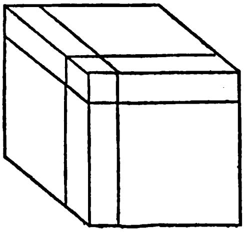
$3ab^2$, by the three following parallepipeds:



and b^3 cubed by the following cube:



All these solid figures combined, form the following figure, which is the cube of $a + b$:



To extract the cube root of a number by means of the formula just established, we reason as follows: Every number, consisting of two or more significant figures, may be considered binomial, and the cube of its tens—that is, the cube of the number on the left of its unit figure—must be a number of thousands; because $10^3 = 1000$, and therefore cannot form any part of the three right-hand figures of the cube. Again, if the part of the root, considered as tens, contains more than one figure, the part on the left of its right-hand figure is a number of tens with regard to that figure, or a number of hundreds, with regard to the first; therefore, as $100^3 = 1000000$, the cube of this part, being a number of millions, cannot form any part of the first six figures of the cube. Continuing to reason thus, it is evident, that in order to find that part of the cube which contains the cube of the left-hand figure of the root, we must *point off the figures of the cube into periods of three figures each*, beginning at the right-hand. This, therefore, is the first step.

Again, the addition of the minor parts of the cube to the cube of the tens can never increase the root of the tens by a unit, because this unit would be 10, and the difference between a^3 and $(a + 10)^3$ is $3a^2 \times 10 + 3a \times 10^2 + 10^3$, which, as 10 is greater than b, is greater than $3a^2b + 3ab^2 + b^3$. Therefore, having pointed off the given number, to find the first figure, or tens of the root, *we extract the root of the greatest cube contained in the left-hand period* of the given number, and place it as we do a quotient figure in division. This is the second step.

We *cube the root, and subtract its cube from the left-hand period of the given number*, and, to the right of the remainder, we *bring down the next period of the cube.* The number thus formed is called *the new dividend.* This is the third step.

Having subtracted the cube of the tens, or a^3, the remaining part of the cube contains $3a^2b + 3ab^2 + b^3$. We must therefore endeavour to find these three parts, add them together, and subtract their sum from the new dividend. Now the root a is a number of tens, and as $3a^2b$ is a number of hundreds, it cannot form part of the two right-hand figures of the new dividend. We therefore *point off those two figures*, and consider the part on the left as containing $3a^2b$; that is, the triple square of a multiplied by b. Then, as the product of two numbers, divided by one of them, gives the other, we *square the root, multiply its square by* 3, *and place the product for a divisor, on the left of the new dividend.* We then *divide the left-hand* *part of the new dividend by the divisor, and place the quotient or unit figure b, in the root,* on the right of a. This is the fourth step.

We next draw a line under the new dividend, and, underneath this, we place the remaining parts of the cube in order, as we find them from the formula. We first *multiply the divisor by b,* and, because b^2 is a number of hundreds, we *place two ciphers on the right,* which evidently gives $3a^2b$, *and place the product under the line.* We then *square b, multiply the square by 3a, and placing a cipher on the right,* because a is a number of tens, we *write this product under the other.* Lastly, we *cube b, and having written its cube under the line, we add the three parts together, and subtract their sum from the new dividend.* As this completes the cube of the two figures of the root already found, it may be called the last step.

If the sum of the three parts should exceed the new dividend, b *is too great,* and must be diminished till the subtraction becomes possible. Also, the subtraction shows whether the given number is, or is not, the exact cube of the root obtained. For, as the sum of the minor parts completes the cube of the root, if this agrees with the new dividend, that is, if the subtraction leaves no remainder, *the given number is the exact cube of the root found.* If the subtraction leaves a remainder, *the given number is not the exact cube.*

379. We will now cube 27, and then, by the formula, extract its root. Thus:

$$27 \times 27 \times 27 = 9 \times 9 \times 9 \times 9 \times 3 = 6561 \times 3 = 19683$$

$$\begin{array}{rl} & \qquad\quad a,b \\ & 19,683\ (2,7 \\ a^3 = & \underline{\ \ 8\ \ }\ \text{greatest cube in 19} \\ \text{partial divisor. } 3a^2 = 12 & \underline{116,83} \\ 3a^2b = & \ \ 84\ 00 \\ 3ab^2 = & \ \ 29\ 40 \\ b^3 = & \ \ \underline{\ \ 3\ 43} \\ & 116\ 83 \end{array}$$

This operation, having been sufficietly explained in the preceding article, needs no farther development. There is, however, a difficulty, too obvious to be neglected, which is, that as the divisor, which is 12 hundreds, is contained 9 times in the 116 hundreds of the new dividend, the scholar would, if not farther assisted, naturally place 9 in the quotient, and would, after completing the minor parts, find the quotient too great. He would next try 8, and, after again performing the operation, would again find the result too great; and, as this difficulty very frequently occurs in extracting the cube root, he would find such useless trials very discouraging. This difficulty we shall endeavour to obviate as follows: The minor parts of the formula being $3a^2b + 3ab^2 + b^3$, if we divide each term by b, we shall have for the general divisor

$$3a^2 + 3ab + b^2$$

Now the second term of this divisor—namely, $3ab$, compared with $3a^2$, is $\frac{3ab}{3a^2} = \frac{b}{a}$; and, therefore, if *we increase the divisor* $3a^2$ *in this ratio, or nearly in this ratio,* we shall, with the divisor so increased, obtain the figure b of the root, in all cases, much nearer the truth than with $3a^2$ alone, and save *the trouble of many useless trials.* For instance, in the above example, we should see how often 12 is contained in 116; we find 9 times: this 9, which should be b, we place over the root already found, which is a, or 20, and thus we have $\frac{9}{20} = \frac{1}{2\frac{2}{9}}$, for the near ratio of b to a. Now as this fraction is greater than $\frac{1}{3}$, we may safely add to the divisor 12 one-third of itself, which is 4. Then, instead of dividing 116 by 12, we divide 116 by 16, and have 7 for the quotient, which is the true figure of the root.

Again, if we extract the root of the cube of 19, as above, we shall find the aforesaid difficulty still increased. Thus:

$$19 \times 19 \times 19 = 361 \times 19 = 6859$$

$$\begin{array}{rrl}
 & a,b & \\
 & 6,859\,(1,9 & \\
 & \underline{1\quad} & \text{the greatest cube in 6.} \\
\text{Partial divisor } 3a^2 = 3 & \underline{58,59} & \text{new dividend.} \\
3a^2b & 27\,00 & \\
3ab^2 & 24\,30 & \\
b^3 & \underline{7\,29} & \\
 & 58\,59 &
\end{array}$$

Here the partial divisor $3a^2$, which is 3 hundreds, is contained in the 58 hundreds of the new dividend 19 times; but having recourse to the ratio $\frac{b}{a}$, we place 19 over a, which is

10, thus, $\frac{18}{10}$; and, as this greatly exceeds a unit, or whole, we find that the remaining part of the general divisor exceeds $3a^2$, and therefore the divisor 3 must be doubled. Wherefore, instead of dividing 58 by 3, we divide it by 6, which gives 9, the true figure sought. This operation, which, after a little practice, will easily be performed by the scholar mentally, will occupy scarcely any time, and save much labour.

380. From what has been said, (378,) we easily perceive that, after pointing off the given number into periods of three figures each, the left-hand figure of the root will be *tens*, if the number contains two periods; *hundreds*, if it contains three; *thousands*, if it contains four, and so on: and, therefore, if any new dividend does not contain the divisor, *we must place a cipher in the root*, bring down a new period to the right of the dividend; place two ciphers on the right of the divisor—*because in squaring the root we should have two ciphers in the product*—and proceed as usual.

Example.

	4,965,203,816 (1706
	1
3	39,65
	21 00
	14 70
	3 43
	39 13
86700	52 203 816
	52 020 000
	183 600
	216
	52 203 816

Observe that, generally, it is only in finding the second figure of the root that we meet with the difficulty mentioned in the preceding article. In the above example, as the double of the divisor 3 is contained $6\frac{1}{2}$ times in 39, we place 7 in the root. The reason for placing rather more in the root, in some cases, will be seen when we consider that the less the figure is which follows a, the less will be the ratio $\frac{b}{a}$; consequently the double of the divisor, which supposes this a ratio of equality, or unit, being somewhat too great, will give the quotient somewhat too little. Having found 17, the two first figures of the root, in seeking a new figure, these two figures become a, or the tens of the root. The triple square of 17, which is 867, is therefore the second divisor. For, as in the square root, in completing and subtracting the minor parts of the cube according to the formula, we have already subtracted the cube of the root found; and, therefore, after the first extraction, *we have continually those parts to repeat.* When to the right of the remainder 52, we have brought down 203, which makes the new divdend 52203, and have pointed off the two right-hand figures, as the divisor 867 is not contained in 522, we place a cipher in the root; and, as the triple square of 170 would evidently be 86700, we also place two ciphers on the right of the divisor 867, and proceed as usual.

381. The scholar may, if he pleases, after finding the second figure of the root, *complete the general divisor* $3a^2 + 3ab + b^2$; in doing which he must observe that, as $3a^2$ is a number of hundreds, $3ab$, a number of tens, and b^2, a number of units; the unit figure of each succeeding part must stand *one place farther towards the right than the unit figure of the preceding.*

Example.

$$
\begin{array}{rl}
 & 4{,}965{.}203{,}816\ (1706 \\
 & \underline{1} \\
\left.\begin{array}{r} 3a^2 = 3 \\ 3ab = 21 \\ b^2 = \underline{49} \end{array}\right\} & 39{,}65 \\
559 & \underline{39\ 13} \\
\left.\begin{array}{r} 3a^2 = 86700 \\ 3ab = 3060 \\ b^2 = \underline{36} \end{array}\right\} & 522038{,}16 \\
8700636 & \underline{522038\ 16} \\
 & {*}{*}{*}{*}{*}{*}{*}{*}
\end{array}
$$

382. We have seen (363) that the square root of a prime

number is impossible, and it is evident that the same mode of reasoning will show that its cube, or any other root, is likewise impossible. We can, however, as in the square root, extend the root decimally, so that the difference between its cube and the given number may be *less than any assignable quantity.* Now, as the root is three times factor in the cube, it is evident that, for every decimal figure in the root, we shall have *three times as many decimals in the cube;* and, therefore, we must constitute three times as many decimal figures on the right of the comma, as we would have decimals in the root. Also it is plain that, as the cube of ,1 is ,001, the cube root of the first three figures on the right of the comma, will give *the tenths,* or first decimal figure of the root; that we must, as in the square root, begin at the comma to separate the decimal part into periods, which must contain three figures each; that the order of each decimal figure in the root may be determined by beginning at the comma, and counting on the periods, tenths, hundredths, thousandths, &c., till we rest on the period which gave it; and, consequently, that *we must place a cipher in the root* when the divisor is not contained in the new dividend, as in integers.

383. To ascertain how far we must extract the root, so that the cube of the root may be within a certain limit, we reason thus: Suppose that we have found the root in integers, and call it a: the part wanting to complete the root, or b, is less than a unit. Let b equal a unit: then, as the cube of the units cannot occupy more than three places of figures, *it cannot affect the cube of the tens,* which is a number of thousands. The only part, therefore, of the cube, which can affect the cube of the tens, is $3a^2b + 3ab^2$. But, as b is a unit, $3a^2b + 3ab^2 = 3a^2 + 3a = (a^2 + a)\ 3$. Wherefore, as b is too great, the difference between the cube of the root found, or cube of the tens, and the true cube, or given number, *cannot exceed* $(a^2 + a)\ 3$, even when the unit b is integral. But, if we extract the root to one place of decimals, b will become $\frac{1}{10}$ if to two places, $\frac{1}{100}$; if to three, $\frac{1}{1000}$, &c.; and, therefore, the part wanting to complete the cube will be *less than* $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. of $(a^2 + a)\ 3$, *accordingly.* If, therefore, having extracted the root in integers, we add this root to its square and multiply the sum by three, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. of this product will show *within what limit we shall find the cube of the root, when it is extracted to one, two, three, &c. places of decimals.*

384. To abridge the operation, we may observe that, as we progress in the extraction of the root, the ratio $\frac{b}{a}$ mentioned above, (379,) is continually decreasing. Thus, if the root contains two figures $\frac{b}{a}$ is less than a unit; if three, it is less than $\frac{1}{10}$; if four, less than $\frac{1}{100}$, &c.; and therefore, if we form the divisor $3a^2$, and place one cipher or figure, on the right of the remainder, instead of three, we may, by division, as in the square root, find *as many new figures of the root as we have already found, minus two figures.*

The following may serve as an illustration. As $\frac{b}{a}$ is the part by which, as we have shown, (379,) the divisor should be increased, in order to give the quotient within a unit, the number by which we are dividing is somewhat too small, and our quotient will, consequently, be somewhat too great; but, we have seen in the example above referred to, that when there are two figures in the root, $\frac{b}{a}$ is less than a unit; when there are three figures, it is less than $\frac{1}{10}$, &c.; hence it is evident that, in general, $\frac{b}{a}$ is less than $\frac{1}{q}$th of the quotient; that is, than a unit of its last, or right-hand figure.

Now, suppose we are dividing 9900 by 100, the quotient being 99, if we diminish the divisor 100 by less than $\frac{1}{99}$ of the quotient,—that is, by less than a unit,—it is plain that the quotient 99 cannot be increased by a unit, and will, therefore, be within the limit proposed.

Analogous reasoning will show that the above abridgement may in all cases be safely applied.

Example.

Suppose we would extract the cube root of 7, so that the cube of the root may differ from 7 by less than a millionth of a unit.

It is evident that the root, in integers, is 1; and, therefore, that $(a^2 + a)\,3 = 6$. Wherefore the cube of the root, with one decimal, will be within $\frac{6}{10}$; that is, within a unit; with two decimals, within $\frac{6}{100}$, or $\frac{1}{10}$ of a unit, &c. of the number 7: consequently, to have the root, so that its cube may be within the proposed limit, we must extract it to one place farther than that limit. The root must therefore contain 7 decimals.

```
                         7, (1,9129311
                         1
   3       )             60,00
   27      }
    81     )
  651                    58 59
  1083     )               1 410,00
    57     }
     1     )
  108871                   1 088 71
  109443     )               321 290,00
     1146    }
        4    )
  10955764                   219 115 28
  10967232     )             102 174 720,00
     51624     }
        81     )
  1097239521                  98 751 556 89
  10977559,2,3)                3 423 163 110 (311
                                 129 895 341
                                  20 119 749
                                   9 142 190
```

The root is therefore 1,9129311, and its cube is 6,9999990913316111107231; which, as was anticipated, differs from 7 by less than a millionth of a unit.

Again, taking the arith. comp. of the part on the right of the successive nines, and placing a cipher on the right, we form a new dividend: and taking for a divisor $3a^2$, as usual, we can, by division, find six more figures of the root; that is to say, two less than the number already found, as follows:

```
1097791618004163) 9086683888927690 (827723
                   304350944894386
                    84792621293554
                     7947208033263
                      262666707235
                       43108383635
                       10174635095
```

It is evident that, as the root, which is now 1,9129311827723, contains thirteen decimals, its cube should, according to the reasoning pursued above, differ from 7, by less than a unit of the twelfth decimal order—that is, by less than a trillionth of a unit—which it does in effect, the actual number being 6,999999999999021854419905611437426582067.

The scholar will see that, by again taking the arith. comp., and proceeding as above, he could now, by division, find twelve more figures of the root, and so on, to any degree of approximation: and that thus, though we can continually approach to the number 7, we never can obtain its exact root.

385. As the cube of a unit is a unit, (170,) the cube root of a fraction must be *a fraction:* for, if its root could be a unit, or more, the cube of this root—that is to say, the fraction itself, would be *a unit or more*, which is absurd. Now, as the fraction, which is the root, is three times factor in the cube, its numerator is (206) three times factor in the numerator of the cube, and its denominator three times factor in the denominator of the cube. Therefore, to find the cube root of a fraction, when both terms are perfect cubes, we have only to find the cube root of each term, and place that of the numerator over that of the denominator. But, as we seldom find both terms perfect cubes, for a general rule, *we reduce the fraction to a decimal, and find the cube root of its decimal value*, by the methods already given. Or, *we multiply the numerator by the square of the denominator, and divide the cube root of the product by the denominator.* This is done upon the supposition that we have multiplied both terms by the square of the denominator, (which does not alter the value of the fraction,) and then divided the cube root of the numerator by that of the denominator, which is the denominator itself.

Also, we shall always find the cube of the root of a fraction within a certain limit, *by extracting the root one place farther than that limit.* This will easily be seen by referring to the reasoning applied, in the preceding article, to the limit of the cube root of 7.

Example.

Suppose we would extract the cube root of $\frac{3}{7}$, so that the cube of the root may differ from $\frac{3}{7}$ by less than the hundred-thousandth of a unit. We extract the root to six places of decimals, thus, (318:)

```
                       ,4̇285 71̇ (,7539
                        343
                        ------
147       ⎫             855,71
 105      ⎬
   25     ⎭
-----
15775                   788 75
---------------------------------
16875      ⎫             66 96428
   675     ⎬
     9     ⎭
-------
1694259                  50 82777
---------------------------------------
1701027      ⎫              1613651571
   20331     ⎬
       81    ⎭
---------
170306091                   15 32754819
---------------------------------------
   17050956,3)                808967520 (47
                              126929268
                                7572574
```

Hence the root is ,753947 +

Or thus:

```
                     3 × 49 = 147
                        147, (5,277
                        125
                        -------
75      ⎫                220,00
 30     ⎬
   4    ⎭
----
7804                     156 08
-------------------------------
8112      ⎫               63 920,00
 1092     ⎬
    49    ⎭
------
822169                    57 551 83
-----------------------------------
833187      ⎫              6 368 170,00
  11067     ⎬
      49    ⎭
--------
83429419                   5 840 059 33
-------------------------------------------
83540187 )                   528 110 670 (632
                              26 869 548
7) 5,277632                    1 807 492
-----------                      136 689
   ,753947 +
```

For the cube root of $\frac{3}{7}$, therefore, we have ,753947 +; the cube of which is ,428570676309809123; and, as this cube agrees with the decimal value of $\frac{3}{7}$, to the fifth decimal inclusive, it is evidently within the limit proposed.

The student should prove the following

Examples.

1. $\sqrt[3]{24389} = 29$
2. $\sqrt[3]{5929741} = 181$
3. $\sqrt[3]{132105539357} = 5093$

4. Required the cube root of 3, so that the cube of the root may differ from 3 by less than a ten-thousandth of a unit.

Answer, 1,44224.

5. Required the cube root of 1729, so that the cube of the root may differ from 1729 by less than a thousandth of a unit.

Answer, 12,002314.

In this example the operation may be abridged by placing 12 in the root, at once, seeing that 1729 only exceeds the cube of 12 by a unit. Also the three last figures, 314, may be found by division.

The student should always find for himself the limit of the root, by the formula $(a^2 + a)3$, and verify it by cubing the root.

6. Required the cube root of 1685176, so that the cube of the root may differ from the number by less than a hundredth of a unit.

Answer, 119,0004.

The reason why this root reaches the limit with so few decimals is, that the next two figures of it are ciphers.

7. Required the cube root of 10795861, so that the cube of the root may differ from the number by less than a thousandth of a unit.

Answer, 221,013648891.

Let the cube root of the following fractions be extracted so that the cube of the root of each fraction may differ from the fraction which gave it by less than the millionth of a unit.

8. $\sqrt[3]{\frac{2}{3}} = ,8735804$
9. $\sqrt[3]{\frac{6}{7}} = ,9499142$
10. $\sqrt[3]{\frac{5}{11}} = ,7688809$
11. $\sqrt[3]{\frac{2}{13}} = ,5358322$
12. $\sqrt[3]{\frac{7}{9}} = ,9196413$
13. $\sqrt[3]{\frac{22}{23}} = ,9852919$

386. As the index of the power to which a quantity is raised shows the number of times that the quantity is factor in that power, it is plain that, when the index is even, the number of factors is even; and, therefore, the power may

be considered as the square of the product arising from the involution of half the number of factors: consequently, if we take half the given index, the quotient will show *to what power the quantity is reduced by taking its square root.* Again, when the index is a multiple of 3, it is plain that the power is the cube of the product arising from the involution of one-third of the number of factors; consequently, if we take one-third of the given index, the result will show *to what power the quantity is reduced by taking its cube root.* Wherefore, in general, if we divide the index of a higher power of a quantity by 2, 3, 4, 5, &c., the result will show *to what power the quantity is reduced by taking its square root, cube root, fourth root, fifth root,* &c.

387. Hence, the roots of all those powers, the indices of which are powers of 2, may be found *by the square root alone.* Thus, to extract the fourth root, that is, to find a quantity from its fourth power, we first extract the square root, which gives the square of the quantity sought: we then extract the square root of the square, which gives the quantity itself. If we would extract the eighth root, the square root of the eighth power gives *the fourth power;* the square root of the fourth power gives *the square,* and the square root of the square, *the quantity itself.*

388. In like manner, the roots of powers, the indices of which are powers of 3, may be found *by the cube root alone.* Thus, the cube root of the ninth power, gives *the cube,* and the cube root of the cube, *the quantity* itself. If we would extract the twenty-seventh root, the cube root of the twenty-seventh power, gives *the ninth power;* the cube root of the ninth power gives *the cube,* and the cube root of the cube, *the quantity sought.*

389. Again, we can, *by the square and cube root combined,* find the roots of those powers, in the indices of which 2 and 3 are the only factors. For example, to find the sixth root, we should extract the square root of the sixth power, which gives the cube; then the cube root of the cube, which gives the quantity sought. Or, first extract the cube root, and then the square root. For the twelfth root, we may first extract the square root, which will give the sixth power; the square root of the sixth power, will give the cube, and the cube root of the cube the quantity sought.

Extraction of the Roots of all Powers.

390. When the index of the power is prime to 2 and 3, and consequently cannot be resolved into those factors, we can no longer obtain the root merely by the square and cube roots, and must, therefore, have recourse to another method.

The scholar has already seen that we obtain the formula for extracting the square or cube root by considering the root under the form of a binomial, which binomial we square or cube accordingly. Now in raising a binomial successively to the higher powers, a law is found to exist in the formation of the terms of those powers, by means of which we obtain a general rule or formula for extracting the roots of all powers.

Let the binomial be $a+b$, as usual. Then if we raise $a+b$ to any power, we shall find that the first term is a, *raised to the given power*, and the last term b, *raised to the same power.* Also, if we subtract a unit from the index of the power, the remainder will show *the number of intermediate terms*, which consist *of ab, the index of a, decreasing, and that of b increasing, each by a unit at a time.* The index of b in the second term, being a unit, is not written. The number of terms, therefore, is *one greater than the index of the power.* Thus, if we would raise $a+b$ to the seventh power, we have, according to the above, the following eight terms: a^7, a^6b, a^5b^2, a^4b^3, a^3b^4, a^2b^5, ab^6, b^7.

391. To have the number which precedes each of the intermediate terms, called *the coefficient*, we observe the following rule: The coefficient of the second term is *the index of the power.* For the coefficient of the third term, *multiply the coefficient of the second term by the index of a in that term, and divide the product by* 2. For the coefficient of the fourth term, *multiply the coefficient of the third term by the index of a in that term, and divide by* 3. In the same manner, to find any coefficient, *multiply the coefficient of the preceding term by the index of a in that term, and divide by the number of terms to that place inclusive.* Thus, for the coefficients of the above series of terms we have

for the second term, 7, the index of the power.

" third " $\frac{7 \times 6}{2} = 21$

" fourth " $\frac{21 \times 5}{3} = 35$

for the fifth term, $\frac{35 \times 4}{4} = 35$

" sixth " $\frac{35 \times 3}{5} = 21$

" seventh " $\frac{21 \times 2}{6} = 7$

The seventh power of $a + b$ is therefore, $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

392. When the power is odd, the number of terms, being one more, is even; and *there are two middle terms, which have each the same coefficient.* The coefficients which succeed the middle ones are *the same as those which precede them, taken in an inverted order.* When the power is even, the number of terms is odd, and there is, consequently, *but one middle term,* the coefficients succeeding which are the same as those which precede it, taken in inverted order. Wherefore, *we do not calculate the coefficients farther than the middle one.*

393. A general formula, for raising a binomial to any power, according to the above method, is obtained as follows: Suppose we would raise $a + b$ to a power signified by any number n. Then we have

$$a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\,.\,3}\,a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4\,.}\,a^{n-4}b^4 \;.\;.\;.\;.\;.\;.\;. + b^n.$$

This general formula, which was discovered by Sir Isaac Newton, and is therefore called *Newton's Binomial Theorem,* furnishes *a general rule for extracting the roots of all powers.*

As in the square and cube roots we obtain the divisor from the second term of the formula, in which b first appears in its simplest form, so, from the second term of the general formula, in which b is still found in its simplest form, *we obtain the divisor for finding b in any root whatever.* In finding the second figure of the root, by means of this divisor, which is na^{n-1}, we are, however, subject to the same kind of difficulty which was experienced in finding the same figure in cube root; and this we shall endeavour to obviate in the same manner as we did in the extraction of that root. Thus, having first divided the second and third terms of the formula by b, to find two terms of a general divisor, we seek, in its simplest form, the ratio of

$\frac{n(n-1)}{2}a^{n-2}b$ to na^{n-1}. To find this, we first multiply both terms by 2, and have $n(n-1)a^{n-2}b$ and $2na^{n-1}$. We then divide both by na^{n-2}, and have $(n-1)b$ and $2a$. The ratio sought is therefore $\frac{(n-1)b}{2a}$. Now to increase the divisor na^{n-1} in a ratio as near to the ratio $\frac{(n-1)b}{2a}$ as we can, we must observe, that b cannot exceed 9, and that when, from the formula, we find that the divisor will be considerably increased, *it is better to diminish b by at least a unit.*

394. To extract the root of any power, therefore, we proceed as follows: As $10^2=100$, $10^3=1000$, $10^4=10000$, $10^5=100000$, &c., it is evident, that in raising to any power, the tens of a binomial root, there will be as many ciphers on the right as there are units in the index of the power; and therefore, to extract any root whatever, it may be shown, as in the square and cube roots, that we must point off the figures of the number *into periods of as many figures each as there are units in the index of the power.* This is the first step.

Seek the greatest figure, which, raised to the given power, is contained in the left-hand period, and place it in the root. This is the second step. This first figure, as well as its required power, may easily be found *by inspecting the table of powers.*

Raise the first figure of the root to the given power, subtract the result from the left-hand period, and to the right of the remainder, bring down the left-hand figure of the next period for a new dividend. This is the third step. The reason for bringing down only the left-hand figure is, that as we raise the root found, which is a number of tens, to the next inferior power for a divisor, we should, as in the square and cube root, have to cut off as many figures on the right as there are units in the index of the given power, minus one; that is, as many, minus one, as are contained in a period.

Then, for a divisor, raise the figure found in the root to the power next inferior to the given power, and multiply the result by the index of the given power. This is the fourth step.

Then in finding the second figure of the root, increase the divisor, as directed above, in order to find b nearer the truth, than by na^{n-1} alone. This is the fifth step, and very important, seeing that to make several useless trials, in very high powers, is no trifling embarrassment.

Next, involve the root to the given power, subtract the result from the two first periods of the given number, and to the right of the remainder bring down the left-hand figure of the next period, to form a new dividend. This may be called the last step. For considering the whole root now found as a, we form the divisor na^{n-1}, and take all the steps in succession as before, except the fifth, which, after the second figure, will not often be needed, observing to subtract the root, when involved, from as many periods as there are figures in the root.

395. With respect to the extraction of the root of a vulgar or decimal fraction, the scholar can find no difficulty, seeing that the methods of operating, for any root whatever, are exactly analogous to those pursued in the square and cube roots. The limit of the approximation of any root may also be found, as in the cube root. Thus, for the fifth root, the formula being $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, and the intermediate terms $5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4$; if we consider $b = 1$, we have, for ascertaining the limit, the formula, $5a^4 + 10a^3 + 10a^2 + 5a$, or $(a^4 + 2a^3 + 2a^2 + a)5$.

Now if, when we have found the root in integers, we involve it, and form a number, according to this formula, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. of the number thus formed, will show within what limit we shall find the fifth power of the root, when extracted to one, two, three, &c. places of decimals.

Again, for the limit of the fifth root of a fraction, suppose the root to equal a unit, which is evidently too great, then as $(a^4 + 2a^3 + 2a^2 + a)5$ is less than $(1 + 2 + 2 + 1)5$, that is, less than 30, for one decimal, the limit will be within $\frac{30}{10}$, or 3 units; for two decimals, within $\frac{3}{10}$; for three, within $\frac{3}{100}$; that is, within $\frac{1}{10}$. Hence, we shall find the fifth power of the root of a fraction within any limit, *by extracting it two places farther than that limit.* In a similar manner, the limit may be found for any other root.

396. The ratio $\frac{(n-1)b}{2a}$ is, in this root, $\frac{4b}{2a}$ or, $\frac{2b}{a}$. Now, if the root contains 2 figures, $\frac{2b}{a}$ is less than 2 units; if the root contains three, it is less than $\frac{2}{10}$; if four, it is less than $\frac{2}{100}$; that is, than $\frac{1}{10}$; consequently, having found a number of figures of the root, we may venture to find, by division, as many more, minus three, as we have already found. As a general rule for all powers, we may always find, by division,

as many more figures as we have already found, minus the number sifinified by $n-1$.

Examples.

1. Required the fifth root of 847288609443.

$$\begin{array}{rll}
 & 84,72886,09443\ (243 & \\
na^4 = 80 & \underline{82} & \\
80 + 48 = 128 & 527 & \\
(24)^5 = & \underline{7962624} & \\
 & \underline{5102620} & \text{new dividend} \\
na^{n-1} = & \underline{1658880} & \text{new divisor, } a \text{ being } 24. \\
(243)^5 = & 847288609443 &
\end{array}$$

Here, na^{n-1} or $na^4 = 80$; and as 80 is contained in 527, six times, we have $b = 6$. But as $\frac{(n-1)b}{2a} = \frac{4b}{2a} = \frac{2b}{a} = \frac{12}{20} = \frac{3}{5}$, we increase 80 by $\frac{3}{5}$ of itself, that is, by $3 \times 16 = 48$, and have 128 for the corrected divisor. This divisor gives 4, the true figure of the root, which, in all probability, would not otherwise be found by the student, till after two useless trials.

2. Required the seventh root of 7974586522691918732936169.

$$\begin{array}{rl}
 & 7974,5865226,9191873,2936169\ (3609 \\
\text{Corrected divisor} & \underline{2187} \\
5103+4080 = 9183) & \underline{57875} \\
 & \underline{78364164096} \\
15237476352000000) & \underline{138170113091918732} \\
 & 7974586522691918732936169
\end{array}$$

In this example, as the divisor 5103 is contained 11 times in 57875, we first assume $b = 9$. Then, as the divisor must, by the formula, be greatly increased, we diminish b by a unit. The ratio $\frac{(n-1)b}{2a}$ or $\frac{6b}{2a} = \frac{3b}{a}$ is accordingly, $\frac{3 \times 8}{30} = \frac{4}{5}$; hence, we increase the divisor 5103 by $\frac{4}{5}$ of itself, which thus becomes 9183. This divisor gives $b = 6$, which is the true figure of the root. As the second divisor, which is 15237476352, is not contained in the new dividend 13817011309, we place a cipher in the root. Now, in forming the new divisor, we have the same significant figures with six ciphers on the right. Also in forming the new dividend, by subtracting $(360)^7$ from three periods, and bringing down another figure, we have the same dividend as before, with seven new figures of the given number on the right. Wherefore, having placed a cipher in the root, instead of forming the last new divisor and dividend, we should evidently find the figure 9 of the root, equally well, either by annexing one more figure to the dividend, or by suppressing one figure of the divisor.

3. Suppose we would have the fifth root of 7, so that the fifth power of the root may differ from 7 by less than a thousandth of a unit. We proceed thus:

$na^{n-1} = 5 \times 1 = 5$	7 (1,47577
$\frac{2b}{a} = \frac{16}{10} = \frac{8}{5}$; then	1
cor. div. $5+8 = 13$	) 60
$(14)^4 \times 5 = 192080$	537824
$\frac{2b}{a} = \frac{1}{9} = 21342$	
new cor. div. 213422	1621760 new dividend
$(147)^5 =$	68641485507
	13585144930 new dividend
$147^4 \times 5 =$	2334744405 " divisor
$(1475)^5 =$	6981682607421875
	183173925781250 dividend
	23666720703125 divisor
	17496880856275 remainder

By the formula $\frac{(n-1)b}{2a}$, which, in this root, is $\frac{2b}{a}$, we correct the first and second divisors, assuming, in each case, $b = 8$, which is, as it should be, a little too great, and obtain, in both cases, the true figure of the root. Also, as the root in integers is 1, we find by the formula $(a^4 + 2a^3 + 2a^2 + a)5$ or $(1 + 2 + 2 + 1)5 = 30$, that the limit of the root will be within $\frac{30}{10}$, or 3 units; for one decimal, within $\frac{3}{10}$; for two, within $\frac{3}{100}$; that is, within $\frac{1}{10}$, for three, &c., and hence, that the root must be extracted two places farther than the limit.

In raising the root successively, to the fifth power, with one, two, three, &c. decimals, which we do in the operation, we shall find the several results to agree with the respective limits established by the formula. Thus, 5,37824, which is the fifth power of 1,4, is within 3 units; 6,864, &c., which is the fifth power of 1,47, is within $\frac{3}{10}$; 6,9816, &c., which is the fifth power of 1,475, is within $\frac{3}{100}$, that is, within $\frac{1}{10}$; and so on for the rest.

Having found the figures 1,4757, we suppress one figure of the divisor, and venture to find, by division, one more figure, which is 7, and, without continuing the operation farther, we raise 1,47577 to the fifth power, and have 6,9999250187372420137021657, which not only differs from 7 by less than a thousandth, as was proposed, but by less than a ten-thousandth of a unit.

4. $\sqrt[5]{281950621875} = 195$

5. $\sqrt[5]{368484741360099} = 819$

6. $\sqrt[7]{678159842212316l6} = 246$

7. $\sqrt[7]{48503336773089989385503320 32} = 9018$

8. Required the fifth root of $\frac{7}{11}$, so that the fifth power of the root may differ from $\frac{7}{11}$ by less than a thousandth of a unit.

Ans. ,91356, the fifth power of which differs from $\frac{7}{11}$ by less than a ten-thousandth of a unit.

9. Required the fifth root of $\frac{2}{3}$, so that the fifth power of the root may differ from $\frac{2}{3}$ by less than a ten-millionth of a unit.

Ans. ,922107911, the difference between the fifth power of which and $\frac{2}{3}$ is ,00000001741400542418646984415334004590586 1156, which is evidently less than a ten-millionth of a unit.

10. Required the seventh root of 13, so that the seventh power of the root may differ from 13 by less than a thousandth of a unit.

Ans. 1,44256. The seventh power of this root being 12,9998158363185256134325318423517 5936, is evidently within the required limit.

SECTION XXI.

OF AN EQUATION—GEOMETRICAL PROPORTION—COMPOUND PROPORTION.

Of an Equation.

397. WHEN two quantities, each consisting of one or more terms, have the sign = between them, the whole expression is called *an equation;* thus, each of the expressions $a = b$, $a + b = c$; $\frac{ad}{b} = c$, $a - c = -b$, $ad = bc$, $\frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1}{3}$, is an equation.

The expression on the left of the sign = is called *the first member;* that on the right, *the second member.* Each quantity contained in either member is called *a term.*

Now, it is plain that, as the members of an equation are equal, we may add to or subtract from each the same or equal quantities: also, that we can multiply or divide each by the same or by equal quantities without disturbing the equation; that is to say, *the results will, in each case, be equal.*

398. Any term, in either member, may be removed (transposed) to the other member, *by merely changing its sign* when so removed. For if it is a positive quantity, it is, when removed, evidently subtracted from the member from which it is taken; and, in placing it in the other member, its subtraction from that member is signified *by placing before it the negative sign.* Again, if it is a negative quantity, we must, in removing it, consider that we have added a quantity equal to it to that member from which it is removed, because such addition (95) would only destroy it; and, in placing it in the other member with the sign +, we signify its addition to that member also. Wherefore, as equals added to equals, or subtracted from equals, must give results which are equal, the equation is not disturbed. Thus, in the equation $\frac{1}{2} + \frac{2}{3} - \frac{5}{6} = \frac{1}{3}$, if we remove the negative quantity $-\frac{5}{6}$ to the second member, and change its sign, we have $\frac{1}{2} + \frac{2}{3} = \frac{1}{3} + \frac{5}{6}$. Again, if we remove $\frac{1}{3}$ to the first member, and change its sign, we have $\frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{5}{6}$, or $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Hence also

it is plain that *all the signs of both members may be changed without disturbing the equation.*

399. An equation is cleared of fractions *by multiplying both members by each denominator successively, or by the least common multiple of all the denominators of those fractions.* Thus, in the above equation, $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, if we multiply both sides by 6, we have $3 + 2 = 5$.

400. When the data of a mathematical question—that is to say, the quantities expressed or implied in its enunciation—are put in equation, the solution of the equation is often the solution of the question.

An equation containing one simple unknown quantity is solved *by bringing all the terms involving the unknown quantity to one side of the equation, and the known quantities to the other;* at the same time, by the above methods, reducing the equation to its simplest form. The unknown quantity being expressed in terms of quantities which are known, thus becomes known.

It is not the province of common arithmetic to treat of this subject at large, but of Universal Arithmetic, or Algebra, to which the student is referred for its full development; it is merely introduced in this place to enable the student more clearly to comprehend some important parts of arithmetic.

Examples.

1. Suppose that into an empty cistern, the capacity of which is 1800 gallons, the water runs during a heavy shower for some time at the rate of 30 gallons per minute; and that, by the increase of the shower, it then runs into it at the rate of 55 gals. per minute, until the cistern is full. During how many minutes must it run at the rate of 55 gals. per minute, that the cistern may be filled in 50 minutes from the commencement of the shower, and what part of the cistern is filled in each interval of time?

Let $x =$ the required number of minutes; then $50 - x$ $=$ the number of minutes during which the water runs at the rate of 30 gals. per minute. Also $55x =$ the number of gallons received by the cistern after the increase of the shower, and $(50 - x)30 =$ the number of gals. previously received; then, $55x + (50 - x)30 = 1800$, or $55x + 1500 - 30x = 1800$, or $25x = 1800 - 1500 = 300$; wherefore, $x = \frac{300}{25} = 12$.

Hence the water runs, during 12 minutes, at the rate of 55 gals. per minute, and consequently during 38 minutes at the rate of 30 gals. per minute; then we have

$55 \times 12 = \ \ 660$ gals. filled after the increase of the shower; also
$30 \times 38 = \underline{1140}$ " before the increase
$\qquad\quad 1800$ whole number of gallons, or content.

consequently, $\frac{66|0}{180|0} = \frac{11}{30} =$ part filled after the increase.

also $\frac{114|0}{180|0} = \frac{19}{30} =$ part filled before the increase.

Proof. $\frac{11}{30} + \frac{19}{30} = \frac{30}{30} = 1$, or the whole cistern.

2. Suppose that a man and his son can together cut 24 cords of wood in 6 days, and that the man alone can do it in 10 days; in how many days can the son alone perform it, and how much wood does each cut per day?

Ans. The son can perform it in 15 days; also, the man cuts $2\frac{2}{5}$ cords, and his son $1\frac{3}{5}$ cords per day.

3. What number is that, the square root of which is double its cube root? Ans. 64.

4. What number is that, of which half the cube root is equal to $\frac{1}{3}$ of its square root? Ans. 11,390625.

Prove this result.

Geometrical Proportion—Rule of Three, or Golden Rule.

401. The ratio of two quantities is often expressed by two points, one above the other, placed between them. Thus, 6 : 3 signifies the ratio of 6 to 3, and is read 6 *is to* 3; $a : b$, signifies the ratio of a to b, and is read a is to b. The scholar already knows that the value of the first of these is $\frac{6}{3} = 2$, and that the value of the last is expressed by $\frac{a}{b}$.

The two homogeneous quantities put in ratio are called *the terms of the ratio;* the first term is called the *antecedent,* and the last the *consequent.* Thus 6 and a are the antecedents, and 3 and b, the consequents of the above ratios.

402. A proportion is *the expression of the relation existing between the terms of two equal ratios.* Thus, 8 : 4 and 6 : 3 form a proportion, each ratio being equal to 2. Four points are placed between the ratios thus—8 : 4 : : 6 : 3, and the proportion is read 8 *is to* 4 *as* 6 *is to* 3. Now it is plain, that as the value of each ratio is 2, the first term bears the same rela-

tion to the second that the third does to the fourth. Whenever, therefore, it may be said of four quantities, that the first is to the second as the third to the fourth, *they are in proportion.* When this cannot be said, *they are not in proportion,* the ratios being unequal. The first and last terms of a proportion are called *the extremes,* and the two middle terms *the means.**

403. Let $a : b :: c : d$ be any proportion whatever; then as $\frac{a}{b} = \frac{c}{d}$, if we multiply both fractions by b, the denominator of the first fraction, which evidently will not disturb the equality, we have $a = \frac{bc}{d}$. Again, if we multiply both sides of the equation by d, we have $ad = bc$; that is to say, in every proportion *the product of the extremes is equal to the product of the means.* This may therefore be called *the fundamental property of a proportion.*

404. We have seen (108) that the product of any two abstract numbers may be considered as representing the content or surface of a rectangular parallelogram. The product of the extremes and the product of the means may therefore be considered as representing two equal rectangles; hence, doubtless, the name *geometrical* has been applied to this kind of proportion, as well as to the ratio upon which it is based.

If any object that, in applying this kind of proportion to the solution of mercantile questions, there would be evident impropriety in supposing a quantity of tobacco to be one side of a parallelogram, and the price of another quantity of tobacco the other; we should remember that this application does not change the fundamental property in the abstract, and that it is *by taking the given numbers in an abstract sense* that we are able to apply the principle to the solution of such questions.

405. When three terms of a proportion are given, the fourth is, by the fundamental property explained above, easily found. For the product of two numbers, divided by either of them, gives *the other;* and as *the product of the extremes is the* same as that of the means, *the product of the extremes, divided by either mean, gives the other mean; also, the product of the means, divided by either extreme, gives the other extreme.* Now, as in all mercantile questions, which are solved by means

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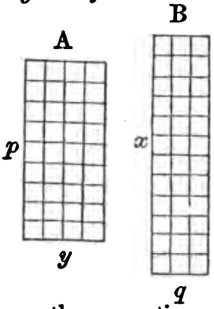
**Les moyens,* (French,) middle terms.

of geometrical proportion, three terms of the proportion are given to find the fourth, the rule for finding this term is called THE RULE OF THREE.

Again, because this rule is the basis of the calculation of Interest, Discount, Barter, Exchange, &c. &c., it is, on account of its extensive application, called THE GOLDEN RULE.

406. If each of two equal products be resolved into any two factors, *those factors are reciprocally proportional;* that is, if A and B be the two equal products, either factor of A is to either factor of B as the remaining factor of B to the remaining factor of A.

As the superficial contents or areas of two equal parallelograms are two equal products, the two factors of each product being the length and breadth of the parallelogram which it represents, if A and B be two equal parallograms, the two dimensions of A are reciprocally proportional to those of B; that is, *either dimension of* A *is to either dimension of* B *as the remaining one of* B *to the remaining one of* A.



Let p and y be the two factors of A; and q and x those of B. Now $py = qx$; divide by y, and we have $p = \frac{qx}{y}$; divide by q, and we have $\frac{p}{q} = \frac{x}{y}$. But $\frac{p}{q}$ is the ratio of p to q, and $\frac{x}{y}$ the ratio of x to y; and these ratios being equal, (402,) we have the proportion $p : q :: x : y$; that is to say, the factors of the two products are reciprocally proportional, those of one product being *the extremes,* and those of the other *the means,* of the proportion. Now py and qx may be any equal products whatever; also p and y any factors whatever of the one, and q and x any factors whatever of the other; wherefore, the demonstration is general.

407. As the above demonstration applies to the parallelograms A and B, it is evident that *the sides which are the dimensions of two equal parallelograms are reciprocally proportional;* those of one parallelogram being the extremes, and those of the other, the means.

408. When the sides which are the dimensions of one parallelogram are reciprocally proportional to those which are the dimensions of another, *the two parallelograms are equal.* For, in this case, the dimensions of the one are the extremes, and those of the other the means of the proportion; and (403) the product of the extremes and that of the means are equal; that is to say, (107) *the areas of the parallelograms are equal.*

We have seen above that either factor of A is to either factor of B as the remaining one of B is to the remaining one of A. Now it is evident, that as A and B are equal, we may change their position, and say either factor of B is to either factor of A as the remaining one of A to the remaining one of B. Wherefore, supposing the factors of A to be p and y, and those of B, q and x, as before, and putting $py=qx$, we have the following proportions:

$$\begin{array}{rll}
 & p:q::x:y & \\
 & p:x::q:y & \textit{alternando} \\
 & y:x::q:p & \\
 & y:q::x:p & \\
\text{putting } qx=py, & q:p::y:x & \textit{invertendo} \\
 & q:y::p:x & \\
 & x:y::p:q & \\
 & x:p::y:q &
\end{array}$$

From the second of these proportions we see that the terms may be taken in alternate order. This is called *alternando*, or *alternately*. From the fifth we see that the terms of the ratios may be inverted, which is called *invertendo*, or *inversely*. Also from an inspection of the whole, we see that in every proportion *we may change the places of the means, or those of the extremes;* also, that *we may take the means for the extremes, and the extremes for the means;* because, in all these changes the extremes are the factors of one product, or the dimensions of one parallelogram, and the means those of the other.

409. Again $p+q:q::x+y:y$. For, if we put $p:q::x:y$ in equation, we have $\frac{p}{q}=\frac{x}{y}$; and if we add a unit to each, which unit is equal to $\frac{q}{q}$ or $\frac{y}{y}$, we have $\frac{p}{q}+\frac{q}{q}=\frac{x}{y}+\frac{y}{y}$, and hence, (202,) $\frac{p+q}{q}=\frac{x+y}{y}$; that is, $p+q:q::x+y:y$. This is called *componendo*, which means that four pro-

portionals are proportional *by composition*; that is, *the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth.*

410. Also, $p-q:q::x-y:y$. For, if instead of adding, we subtract a unit from each member of the equation, $\frac{p}{q}=\frac{x}{y}$, we have $\frac{p}{q}-\frac{q}{q}=\frac{x}{y}-\frac{y}{y}$; and hence, $\frac{p-q}{q}=\frac{x-y}{y}$; that is, $p-q:q::x-y:y$. This is called *dividendo*, which means that four proportionals are proportional *by division*; that is to say, *the difference between the first and second is to the second, as the difference between the third and fourth is to the fourth.*

411. Things which are equal to the same thing are equal to one another; therefore, two *ratios which are each equal to the same ratio are equal to one another.* Now, (409) as

$p+q:q::x+y:y$ alternando
$p+q:x+y::q:y$
But, (410) $p-q:q::x-y:y$ and alternando
$p-q:x-y::q:y$

Hence, as the ratios $p+q:x+y$ and $p-q:x-y$ are both equal to $q:y$, they are equal to one another; consequently, $p+q:x+y::p-q:x-y$; and alternando, $p+q:p-q::x+y:x-y$; that is to say, when four quantities are proportional, *the sum of the terms of the first ratio is to their difference, as the sum of the terms of the last ratio to their difference.*

412. Again, supposing p greater than q, as (411)
$p-q:x-y::q:y$, and (408)
$q:y::p:x$; then (411)
$p-q:x-y::p:x$ alternando (408)
$p-q:p::x-y:x$ and invertendo
$p:p-q::x:x-y$; that is to say, *the first is to its excess above the second, as the third is to its excess above the fourth.*

413. If we put the proportion $p:q::x:y$ in equation, thus: $\frac{p}{q}=\frac{x}{y}$, because (397) the equality is not disturbed by multiplying or dividing both sides by any number, $\frac{2p}{q}=\frac{2x}{y}$; $\frac{3p}{q}=\frac{3x}{y}$, &c.; also $\frac{p}{2q}=\frac{x}{2y}$; $\frac{p}{3q}=\frac{x}{3y}$ &c., and hence (406) it is

evident that $2p : q :: 2x : y$
$3p : q :: 3x : y$, &c., also that
$p : 2q :: x : 2y$
$p : 3q :: x : 3y$, &c.,
from which we infer, that in any proportion *the antecedents may be both multiplied or both divided by any number without disturbing the proportion*, and that it is the same with the consequents. Also seeing that magnitudes have the same ratio to one another which their equimultiples have, that
$2p : 3q :: 2x : 3y$
$3p : 4q :: 3x : 4y$, &c., and therefore, *the antecedents may be both multiplied or both divided by one number, and the consequents by another* without disturbing the proportion; that is $\frac{p}{100} : 40q :: \frac{x}{100} : 40y$, &c. Again, alternando, as $2p : 2x :: 3q : 3y$
$3p : 3x :: 4q : 4y$, &c., it is plain that *the terms of each ratio may be both multiplied or both divided by any number, or the terms of one multiplied and those of the other divided by any number, without disturbing the proportion.*

414. If we have any two proportions
$a : b :: c : d$ and
$p : q :: x : y$, *we may add them together term for term, subtract one from the other, term for term, also multiply or divide one by the other, term for term, without disturbing the proportion.* For $\frac{a}{b} = \frac{c}{d}$, and $\frac{p}{q} = \frac{x}{y}$; and adding equals to equals, $\frac{a}{b} + \frac{p}{q} = \frac{c}{d} + \frac{x}{y}$; then reducing to a common denominator, and multiplying both sides by that denominator, we have $(a+p)d + (a+p)y = (b+q)c + (b+q)x$; or $(a+p)(d+y) = (b+q)(c+x.)$ Then, as the factors are reciprocally proportional, (406,) $a+p : b+q :: c+x : d+y$.

Again, as $\frac{a}{b} = \frac{c}{d}$ and $\frac{p}{q} = \frac{x}{y}$, taking equals from equals, we have $\frac{a}{b} - \frac{p}{q} = \frac{c}{d} - \frac{x}{y}$; and reducing and multiplying as before, we have $(a-p)d - (a-p)y = (b-q)c - (b-q)x$, or $(a-p)(d-y) = (b-q)(c-x,)$ and as the factors are reciprocally proportional, we have $a-p : b-q :: c-x : d-y$.

Again, as $\frac{a}{b}=\frac{c}{d}$ and $\frac{p}{q}=\frac{x}{y}$ (397) $\frac{a}{b}\times\frac{p}{q}=\frac{c}{d}\times\frac{x}{y}$; that is, $\frac{ap}{bq}=\frac{cx}{dy}$, and hence (402) $ap : bq :: cx : dy$.

Also, as $ad=bc$ and $py=qx$, (403,) it is plain, that $ad : bc :: py : qx$, and alternando, $ad : py :: bc : qx$, and putting this in equation, we have $\frac{ad}{py}=\frac{bc}{qx}$, or $\left(\frac{a}{p}\right)\left(\frac{d}{y}\right)=\left(\frac{b}{q}\right)\left(\frac{c}{x}\right;)$ then, as the factors are reciprocally proportional, $\frac{a}{p}:\frac{b}{q}::\frac{c}{x}:\frac{d}{y}$.

We have therefore added, subtracted, multiplied, and divided the given proportions, term for term, without disturbing the proportion, as was proposed.

415. As the first term of a proportion has the same relation to the second that the third has to the fourth, if the first is equal to the second, the third must be equal to the fourth; if the first is greater than the second, the third must be greater than the fourth, and if the first is less than the second, the third must be less than the fourth. Therefore, inversely, *if the fourth term is greater than the third, equal to it, or less, the second must be greater than the first, equal to it, or less, accordingly.*

416. We have already observed that in many arithmetical questions, three terms of a proportion are given to find the fourth. *Two of the three given quantities are of the same kind, and the remaining one is of the same kind as the answer.*

To solve a question of this nature, *we write the quantity, which is of the same kind as the answer, for the third term,* with the sign as : : before it; that is, on the left of it. Then, from the reading of the question, we determine *whether the answer or fourth term should be greater or less than the term written;* if the answer should be greater, *the greater of the two remaining terms must be the second;* but, if the answer should be less, *the greater of the two remaining terms must be the first.*

Having arranged the three terms, we divide the product of the means by the given extreme, which (405) gives the extreme sought, or answer.

Examples.

1. If 11 bushels of corn cost $5, how much must I pay for 13 bushels at the same rate?

$$\begin{array}{l} \overset{\text{bush.}}{11} : \overset{\text{bush}}{13} :: \overset{\$}{5} \\ \underline{5} \\ 11)\overline{65} \\ \$5{,}90\tfrac{10}{11},\ \text{or}\ \$5{,}\,91\ \text{—} \end{array}$$

Questions of this nature may be said to consist of two parts, one of which is a supposition, and the other a demand.

Now, it is *from the demand* that we discover the nature of the answer, or fourth term. Thus, in the above example, the demand is, how much (money of course implied) must I pay? We therefore place $5, the term which is of the same kind as the answer, in the third place, thus— : :, $5; then, *from a comparison of the supposition with the demand,* we easily see that the answer must be greater than $5; because 13 bushels must cost more than 11, at the same rate; we therefore place 13 for the second term and 11 for the first. Then, by multiplying 13 and 5, which are the two means, we have 65, which (403) is also the product of the two extremes, and, dividing by 11, the given extreme, we have $5,90$\tfrac{10}{11}$, or $5,91 — for the other extreme, which is the fourth term of the proportion, or answer.

2. Having paid $5,90$\tfrac{10}{11}$ for 13 bush. of corn, how much should I pay for 11 bushels?

$$\begin{array}{l} 13 : 11 :: \overset{\$}{5{,}90}\tfrac{10}{11} \\ \phantom{13 : 11 :: 5{,}9}\underline{11} \\ 13)\ 65{,}00\ (\$5\ \text{Answer.} \\ \underline{65} \\ {*}{*} \end{array}$$

3. If for $5,90$\tfrac{10}{11}$, we have 13 bushels of corn, how much should we have, at the same rate, for $5?

$$\$5{,}90\tfrac{10}{11} = \$\tfrac{65}{11};\ \text{then,}$$

$$\frac{65}{11} : \frac{5}{1} :: \frac{13}{1},\ \text{and}\ \frac{\cancel{5}}{1} \times \frac{\cancel{13}}{1} \times \frac{11}{\cancel{65}} = \overset{\text{bush.}}{11.}\ \ \text{Answer.}$$

Having expressed all the terms fractionally, we signify the multiplication of the two means by placing the sign × between them. Then, to divide by the first term, (213,) we multiply by its reciprocal.

Thus a Rule of Three question is reduced to one multiplication of fractions; and, as this affords great opportunity for cancelling, it is perhaps, in general, the very best mode of operating.

4. If I have paid \$5 for 11 bush. of corn, how much should I have for $\$5{,}90\frac{10}{11}$?

$$\$5{,}90\tfrac{10}{11} = \$\tfrac{65}{11}\text{; then, } \$\tfrac{5}{1} : \$\tfrac{65}{11} : : \overset{\text{bush.}}{\tfrac{11}{1}} \text{ and}$$

$$\frac{\cancel{6}\cancel{5} \times \cancel{1}\cancel{1} \times 1}{\cancel{1}\cancel{1} \times \cancel{1} \times \cancel{5}} = 13. \text{ Answer.}$$

The scholar may accustom himself to make similar changes in any question solved by proportion. Also, the several operations will serve as proofs of each other. For further proof, he may *put the proportion in equation, or find the product of the extremes and that of the means, which* (403) *must be equal.*

417. When, by the method pursued in the preceding article, we have assigned each of the three terms its proper place in the proportion, though we consider the two first terms as forming the first ratio, it is evident that in reality *they do not form a ratio with each other, until reduced to units of the same magnitude, or to fractions of the principal unit,* in which state we can operate upon them (324) as upon abstract numbers. For example, suppose that we pay 3 d. for half a peck of apples; what quantity can we buy at the same rate for £5 sterling?

$$\overset{\text{d.}}{3} : \overset{\text{£.}}{5} : : \overset{\text{peck.}}{\tfrac{1}{2}}$$

Having arranged the terms, we say, as 3 d. is to £5, so is the quantity bought for 3 d. to the quantity that can be bought for £5, which is doubtless correct. Now the value of the ratio 3 : 5, if the quantities were alike, would be $\frac{3}{5}$; but this is evidently not the ratio of 3 d. to £5. Wherefore, we must reduce £5 to pence, or 3 d. to the fraction of a pound, in order to have the true ratio. Reducing £5 to pence, therefore, we have £5 = 1200 d. Then, as $\frac{1}{2}$ peck is $\frac{1}{8}$ of a bushel, we have

$\overset{\text{d.}}{3} : \overset{\text{d.}}{1200} : : \overset{\text{bush.}}{\tfrac{1}{8}}$ or, (413) multiplying the two antecedents by 8, and dividing the two terms of the first ratio by 12, we have

$$\overset{\text{d.}}{2} : \overset{\text{d.}}{100} : : \overset{\text{bush.}}{1} : \overset{\text{bush.}}{50}. \text{ Answer.}$$

Again, by reducing 3 d. to the fraction of a pound, we have 3 d. $= \frac{3}{240} = £\frac{1}{80}$. Then,

$\overset{\text{£.}}{\frac{1}{80}} : \overset{\text{£.}}{\frac{5}{1}} :: \overset{\text{bush.}}{\frac{1}{8}}$. Then (413) multiplying the two antecedents by 80, we have

$$\overset{\text{£.}}{1} : \overset{\text{£.}}{5} :: \overset{\text{bush.}}{10} : 50 \text{ bush. Answer.}$$

418 The product of the two means, divided by the given extreme, is *the fractional representation of the fourth term or answer.* Wherefore, after establishing the true ratio between the two first terms, because the two means are the two factors of the numerator, and the given extreme the divisor or denominator, it is plain that upon the principle of cancelling, we may, without affecting the answer, *divide either mean by the given extreme, or the given extreme by either mean, or either mean and the given extreme both by the same number.* Also, that *either mean and the given extreme may be both multiplied by any number;* because this is only multiplying the two terms of the fractional result, each by the same number, which does not alter its value.

419. If, after establishing the true ratio of the two first terms, we would, to facilitate the operation, reduce the third term to a different form, as it is a factor of the numerator, it is evident (165) that in operating upon this term, *we operate directly upon the answer*, or fourth term; that is to say, if we multiply or divide the third term by any number, *we multiply or divide the answer by the same number;* or, if we reduce the third term to any denomination, *the answer will be of the same denomination*, the terms of the first ratio being considered abstract

Examples.

1. If $6\frac{3}{4}$ yds. of cloth cost 13 s. 4 d., how much will 81 yds. cost, at the same rate?

$$6\tfrac{3}{4} = \tfrac{27}{4}; \text{ then } \begin{array}{cccccc} \text{yds.} & & \text{yds.} & & \text{s.} & \text{d.} \\ \frac{\cancel{27}}{\cancel{4}} & : & \cancel{81} & :: & 13 & 4 \\ & & 12 & & & 12 \\ \hline & & £8 & & 0 & 0 \end{array}$$

We here divide the second term by the first, to do which, we first observe that 81 contains 27 three times; then, as it is plain that the fourth part of 27, or $\frac{27}{4}$, should be contained in 81 just 4 times as often as the whole 27, we say, that $\frac{27}{4}$ is contained 4 times 3, or 12 times; we therefore write 12 under 81, and multiply 13 s. 4 d. by 12, which gives £8 for the answer.

Hence, we infer that if a whole number is divisible by the numerator of a fraction, *it is divisible by the fraction ;* for, to obtain the quotient, we have only to multiply the result by the denominator of the fraction.

2. If I pay £8 for 81 yds., how many can I buy for 13 s. 4d.?

13 s. 4 d. $=$ £$\frac{2}{3}$; then

£. £. yds. yds.

$$\underset{12}{\cancel{8}} : \frac{\cancel{2}}{\cancel{3}} : : 81 : 6\tfrac{9}{12},$$ or $6\frac{3}{4}$ yds. Answer.

3. If, for 13 s. 4 d., we have $6\frac{3}{4}$ yds., how many can we buy with £8 ?

$6\frac{3}{4} = \frac{27}{4}$ and 13 s. 4 d. $=$ £$\frac{2}{3}$; then

£. £. yds.

$$\frac{\cancel{2}}{3} : \cancel{8} : : \frac{27}{\cancel{4}} : 81$$ yards. Answer.

Here, as the first term $\frac{2}{3}$ should be inverted, the denominator 3 is a multiplier; wherefore, as the two divisors 4 and 2 cancel 8, we have $3 \times 27 = 81$, the answer.

4. If 81 yds. cost £8, how much should we pay for $6\frac{3}{4}$ yds.?

yds. yds. £. £. £.

$$\underset{12}{\cancel{81}} : \frac{\cancel{27}}{\cancel{4}} : : 8 : \tfrac{8}{12} = \tfrac{2}{3} = 13\text{ s. } 4\text{ d.}$$ Answer.

420. The length and breadth of the floor or wall of a room of regular form, and the length and breadth of carpet, paper, or other material which is to cover it, being evidently the dimensions of two equal parallelograms, are (406) reciprocally proportional; consequently, if we wish to find the length of carpet, of a given breadth, that will cover a floor of given dimensions, *the breadth of the carpet is to either dimension of the floor as the remaining dimension of the floor to the length of the carpet* or quantity sought. The dimensions of the floor are therefore the means, and their product, or area of the floor, being also the area, that is, the product of the length and breadth, of the carpet, must when divided by the given extreme, or breadth of the carpet, give its length, which is the other extreme.

Now, the breadth of the carpet is generally referred *to the yard*, as the standard unit, and the dimensions of the floor *to to the foot*; wherefore, if we put $\frac{w}{d}$ for the fractional width of carpet in yards, or general divisor, and l and b for the dimensions of the floor in feet, $\frac{l}{3}$ and $\frac{b}{3}$ will be the same dimensions in yards. Then, inverting the divisor, as usual in proportion, we shall have for the general formula $\frac{d}{w} \times \frac{l}{3} \times \frac{b}{3}$, or, $\frac{bdl}{9w}$ = length of carpet in yards.

421. As either mean may be divided by the given extreme, we may *divide either the length or breadth of the floor by the given breadth of carpet,* when this can be done, *taking care to reduce both to the same name, and multiply the quotient by the other dimension, for the required length of carpet in feet.* In this last case, the quotient is *the number of breadths* of carpet, and the remaining dimension *the length of each,* which may remain in feet or be reduced to yards at option. Also it is evident, that in this case only can the floor be carpeted with an exact number of breadths, and consequently without waste.

When practicable, it is of course best to express both dimensions of the floor in yards.

Examples.

1. How many yards of carpeting, ¾ yd. wide, will cover a floor 27 ft. long by 18 ft. broad?

$\overset{\text{yd.}}{\tfrac{3}{4}} : \overset{\text{ft.}}{18} :: \overset{\text{ft.}}{27}$, or $\overset{\text{yd.}}{\frac{\cancel{3}}{\cancel{4}}} : \overset{\text{yds.}}{\frac{\cancel{6}}{8}} :: \overset{\text{yds.}}{9} : 72$ yds. Answer.

Or, $\overset{\text{yd.}}{\frac{\cancel{3}}{\cancel{4}}} : \overset{\text{yds.}}{\frac{\cancel{6}}{12}} :: \overset{\text{yds.}}{6} : 72$ yds. Answer, as before.

Here, taking the room lengthwise, we have 8 breadths, each 9 yds. long. Or, taking it crosswise, we have 12 breadths, each 6 yds. long.

2. How many yards of carpeting, 1¼ yds. wide, will cover a floor 45 ft. long by 23 ft. broad?

$\overset{\text{yds.}}{\tfrac{5}{4}} : \overset{\text{ft.}}{45} :: \overset{\text{ft.}}{23}$, or $\overset{\text{yds.}}{\tfrac{5}{4}} : \overset{\text{yds.}}{15} :: \overset{\text{yds.}}{\tfrac{23}{3}}$, and $\frac{4}{\cancel{5}} \times \frac{\cancel{15}}{1} \times \frac{23}{\cancel{3}} = 92$ yds. Answer.

Here, as $\frac{5}{4}$ is contained in 15 just 12 times, there will be 12 breadths, each 23 ft., or 7 yds. 2 ft. long each. Also, the same carpet could not, in the other direction, be applied without waste.

422. Another problem of some importance is, to ascertain the breadth of carpet, approximate to a given breadth, that will, without waste, cover a floor of any given dimensions.

First, *divide either dimension of the floor by the given breadth of carpet reduced to feet, for the approximate number of breadths; then taking the nearest whole number, divide the same dimension by the number of breadths, which will give the required approximate breadth of carpet.*

Example.

What breadth of carpet, nearly a yard wide, will, without waste, cover a floor 19 ft. by 13 ft.? and how many yards will it take?

First, $\frac{13}{3} = 4\frac{1}{3}$; then, taking 4 for the number of breadths, $\frac{13}{4} = 3\frac{1}{4}$ ft., or 3 ft. 3 in. for the required breadth of carpet. Hence, there will be 4 breadths, each 19 ft., or $6\frac{1}{3}$ yds. long, and $4 \times 6\frac{1}{3} = 25\frac{1}{3}$ yds. the required length. Or, as $\frac{13}{4}$ ft. $= \frac{13}{12}$ yds., thus:

$\overset{\text{yds.}}{\frac{13}{12}} : \overset{\text{yds.}}{\frac{13}{3}} :: \overset{\text{yds.}}{\frac{19}{3}}$; Or, (189) $\overset{\text{yd.}}{1} : \overset{\text{yds.}}{4} :: \overset{\text{yds.}}{\frac{19}{3}} : \overset{\text{yds.}}{\frac{76}{3}} = 25\frac{1}{3}$, as before.

Again, $\frac{19}{3} = 6\frac{1}{3}$, and, taking 6 for the number of breadths, we have $\frac{19}{6} = 3\frac{1}{6}$ ft., or 3 ft. 2 in. for the required breadth of carpet; that is, there will be 6 breadths, each 13 ft. or $4\frac{1}{3}$ yds. long, and $6 \times 4\frac{1}{3} = 26$ yds. the required length. Or, as $\frac{19}{6}$ ft. $= \frac{19}{18}$ yds., thus:

$\overset{\text{yds.}}{\frac{19}{18}} : \overset{\text{yds.}}{\frac{19}{3}} :: \overset{\text{yds.}}{\frac{13}{3}}$, Or (189) $\overset{\text{yd.}}{1} : \overset{\text{yds.}}{6} :: \overset{\text{yds.}}{\frac{13}{3}}$, or *$\overset{\text{yd.}}{1} : \overset{\text{yds.}}{2} :: \overset{\text{yds.}}{13} : 26$ yds., as before.

From which we infer, that if we would carpet the floor lengthwise, with carpet nearly yard wide, we must choose a carpet 3 ft. 3 in., or rather more than 1 yd. 0 qr. 1 n. wide, of which it will require 4 breadths, each 6 yds. 1 ft. in length, making $25\frac{1}{3}$ yds. Also, if we would carpet it crosswise, we must choose a carpet 3 ft. 2 in., or rather less than 1 yd. 0 qr. 1 n. wide, of which we require 6 breadths, each 4 yds. 1 ft. in length, making 26 yards.

Proof.

78 ft. × $3\frac{1}{6}$ ft. $= 247$ ft. area of last carpet.
76 ft. × $3\frac{1}{4}$ ft. $= 247$ ft. " first carpet.
13 ft. × 19 ft. $= 247$ ft. " floor.

423. Having seen (402) that when the measures or sides of two equal parallelograms are placed in reciprocal proportion, those of the one become the means, and those of the other the extremes; it follows that when the area of any given parallelo-

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*See Article 210.

gram is divided by one side of another parallelogram, of equal area*, the quotient will be *the other side of that other parallelogram.* We must; however, observe that the given area and given side must be of the same name; that is, if the given area is expressed in square yards, the length of the given side must be expressed in yards; if the given area is expressed in square feet, the length of the given side must be expressed in feet, &c. Also, that the given area must be *a multiple of the given side,* if we wish the result to be a whole number.

Suppose that from one side of a large field we would cut off a piece of ground of a given breadth, say 120 ft., which shall contain just an acre.

The area of an acre in s. f. is 43560, and $\frac{4356|0}{12|0} = 363$ feet, the required length.

Or, to ascertain whether the length will be a whole number, we may resolve 43560 into its prime factors $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 11 \times 11$, and then see, if by the involution of any of these factors, we can form 120. We find that $2 \times 2 \times 2 \times 3 \times 5 = 120$; consequently, the product of the other factors $3 \times 11 \times 11 = 363$ ft. is the required length.

424. When the first of three given numbers is to the second as the second is to the third, we say that the numbers are *in continual proportion.* The third is called *a third proportional to the two first,* and the second *a mean proportional between the first and third.* When we say, as the first is to the second, so is the second to the third, the second, being both antecedent of the last ratio, and consequent of the first, forms of itself the two means, the product of which is the square of the second; consequently, (405,) to find a third proportional to two given numbers, *we square the last and divide by the first.* Or, which is the same thing, *we multiply the last by its ratio to the first.* We must, however, observe, that the second must be *a multiple or a measure of the first,* or both must be composed of powers of the same prime numbers, otherwise (179) the third proportional cannot be a whole number.

Thus, to find a third proportional to 2 and 6, we say $\frac{6 \times 6}{2} = 18$, the number sought. For a third proportional to 8 and 9, which are prime to each other, we have $\frac{9^2}{8} = \frac{81}{8} = 10\frac{1}{8}$, Answer.

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*We are here speaking of right-angled (rectangular) parallelograms.

425. As in the preceding article, the product of the extremes is equal to the product of the means; the product of the first and third is equal to the square of the second; wherefore, to find a mean proportional between two numbers, *we take the square root of their product.*

Thus, to find a mean proportional between 3 and 27, we say, $3 \times 27 = 81$, and $\sqrt{81} = 9$, the required mean.

426. Let A, B and C, be any three numbers, the ratio of A to B is $\frac{A}{B}$, and that of B to C is $\frac{B}{C}$. Now, $\frac{A}{B} \times \frac{B}{C} = \frac{A}{C}$; that is, *the ratio of* A *to* C *is compounded of the ratios* A *to* B *and* B *to* C.

427. When the numbers are in continual proportion $\frac{A}{B} = \frac{B}{C}$; and therefore $\frac{A}{B} \times \frac{A}{B} = \frac{A}{C}$, or, $\frac{A^2}{B^2} = \frac{A}{C}$. Hence also $\frac{B^2}{C^2} = \frac{A}{C}$.

This is called *duplicate ratio.* Therefore, when three numbers are in continual proportion, the ratio of the first to the third is duplicate of the ratio of the first to the second, or of the second to the third; that is to say, *it is the square of either of those ratios.*

428. When there are four numbers A, B, C, D, the ratio of A to D is compounded of the ratios A to B, B to C, and C to D. Thus $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{A}{D}$; and, in general, whatever may be the number of quantities, *the ratio of the first to the last is compounded of the ratios, the first quantity to the second, the second to the third, the third to the fourth, and so on to the last.*

429. When four numbers are in continual proportion, as the ratio of the first to the fourth involves three equal ratios, it is called *the triplicate ratio or cube* of any one of those ratios; when there are five numbers, the ratio of the first to the fifth is *the quadruplicate or fourth power* of any one of those ratios; when there are six numbers, the ratio of the first to the sixth is *quintuplicate* of the ratio of the first to the second, &c.; and in general, when a series of numbers is in continual proportion, the ratio of the first to the last is equal to *the ratio of the first to the second raised to a power signified by the number of terms, less one.*

430. The astronomer Kepler found that *the squares of the times in which any two planets revolve round their primary*

are to one another as the cubes of their distances from that primary; wherefore, if we know the times of the revolution of any two planets in the solar system, and the distance of one of them from the Sun, the distance of the other may be found by a simple proportion, and by extracting the cube root; or, if we know the distance of each from the Sun, and the time of the revolution of one of them, the time of the revolution of the other may be found by a proportion, and by extracting the square root.

Examples.

1. Let the time of the revolution of the planet Mercury about the Sun be 87 d. 23 h. 14 m. 32,7 sec.; that of the revolution of Venus 224 d. 16 h. 41 m. 27,5 sec.; also the distance of Venus 68514044 miles; required the distance of Mercury from the Sun.

Square of revol'n. of Venus in seconds. Square of revol'n. of Mercury in seconds.
As 376891262347656,25 : 57767185263445,29
Cube of distance of Venus. Cube of distance of Mercury.
: :321616859411343801397184 :49295121849560890257289,

and extracting the cube root of this last term, we have 36666375,479 + miles, the required distance of Mercury from the Sun.

2. Let the time of the revolution and distance of the planet Venus be as in the above example; also the time of the revolution of the Earth 365 d. 5 h. 48 m. 51 sec.; required the distance of the Earth from the Sun.

Ans. 94719068,71 + miles.

3. Let the time of the revolution of the Earth, also the time of the revolution and the distance of the planet Mercury, be as in the above examples; required the distance of the Earth from the Sun. Ans. 94719068,71 + miles.

4. Let the time of revolution and the distance of the Earth, also the distance of Mercury, be as above; required the time of the revolution of Mercury.

Ans. 87 d. 23 h. 14 m. 32 + sec.

Compound Proportion.

431. The name Compound Proportion may properly be applied *to every proportion in which several ratios are compounded, that is, multiplied together.*

This rule is frequently, and we think improperly, called the Double Rule of Three, or Rule of Five; because, though in many questions five terms are given, in others we find seven, nine, eleven, &c.

In common with Simple Proportion, *one term is given, which is of the same kind as the answer;* and of the rest, *two and two are of the same kind;* consequently, each couple forms *a ratio.*

432. To solve a question: First, *write the term which is like the answer by itself; then arrange the two terms of each ratio as in Simple Proportion;* that is, exactly as if the result depended on them alone.

Having arranged all the terms, cancel the antecedent and consequent of any ratio, or the antecedent of any one and consequent of any other, not forgetting the term first written, which may be cancelled with any of the antecedents, as in Simple Proportion. Lastly, multiply the remaining consequents and term first written together, and divide the result by the product of the remaining antecedents.

Examples.

1. If 40 masons can build a wall 1350 feet long, 3 ft. thick, and 12½ ft. high, in 9 days, when the days are 10 hours long, how many masons will build a wall 5400 ft. long, 2½ ft. thick, and 15 ft. high, in 15 days, when the days are only 8 hours long?

Ant's.		Cons's.		Masons.
$\cancel{1350}$		$\cancel{5400}$	4	$\cancel{40}$
$\cancel{3}$		$\cancel{2\frac{1}{2}}$		
$\cancel{12\frac{1}{2}}$	$\cancel{5}$	$\cancel{15}$		
$\cancel{15}$		$\cancel{9}$	3	
$\cancel{8}$		10		

As the question requires masons, we write 40, the given number of masons, by itself. Then, for the ratios, we say 5400 ft. will require more than 1350 ft.; wherefore, we write 1350 for the antecedent and 5400 for the consequent; we also accompany each of these with its respective thickness and height. We then say, it will require fewer men to build the wall in 15 d. than in 9 d.; consequently, we write 15 for the antecedent and 9 for the consequent. Lastly, as it will require more men when the days are only 8 hours long, we write 8 for the antecedent and 10 for the consequent.

We then find that 1350 cancels 5400 and gives 4 for the quotient, which we write opposite 5400. Then 2½ cancels 12½, giving 5 for the quotient; and 8 times 5 cancels 40.

Then 15 cancels 15, and 3 cancels 9, and gives 3. Lastly, 4 . 3 . 10 = 120, the number of masons required.

2. If, when money is loaned at 6 per cent. per annum, A borrows of B £26 for 120 days, how long, in order to cancel the obligation, must A lend B £17 6 s. 8 d., when money is loaned at $4\frac{1}{2}$ per cent.? Ans. 240 days.

3. If, when money is at 5 per cent., B borrows of A $840 for 8 mos., how long must A retain $636,36$\frac{4}{11}$ of B's money, to cancel the obligation, when money is at $7\frac{1}{2}$ per cent.? Ans. 7 mos. $1\frac{1}{5}$. d.

4. If 7 horses eat 231 bushels of oats in 132 days, how many days will 561 bush. serve 11 horses, at the same rate? Ans. 204 days.

5. If a troop of horse march 506 miles in 23 days of 10 hours, in how many days of 8 hrs. will it march 704 miles, at the same rate? Ans. 40 days.

6. If two men mow 9 acres of grass in 3 days, working 8 hrs. per day, how many men will mow 27 acres in 4 days, working 9 hrs. per day? Ans. 4 men.

7. If a garrison of 1200 men is victualled for 6 mos. of 30 days, so as to allow each man 2 lbs. weight per day, how much must each man be allowed per day to make the provisions last 8 mos., supposing it reinforced with 300 men, bringing with them a supply of 108000 lbs.? Ans. $1\frac{1}{2}$ lbs.

8. A garrison of 1000 men, victualled for 8 mos. of 30 days, at the daily rate of $2\frac{1}{2}$ lbs. per man, is besieged 40 days; but, making a sortie, they drive off the enemy; now suppose that, having lost 200 men, they are immediately reinforced with 400, who bring 40000 lbs. of provisions, what must each man's daily ration be to make the provisions last? Ans. $2\frac{1}{4}$ lbs.

SECTION XXII.

POSITION—ALLIGATION—PERMUTATION — ARRANGEMENT — COMBINATION.

Position.

433. Position, or Supposition, is that branch of arithmetic which finds a number sought from a number, called the *supposition*, which is arbitrarily assumed, and is operated upon, ac-

cording to the conditions of the question, exactly as though it was the true number.

Position is called Single or Double, accordingly as the result is obtained from one or two suppositions. Those questions the data of which require only that the quantity operated upon shall be increased or diminished in a certain ratio or by quantities which are aliquot parts or multiples of itself, belong to Single Position, and may be solved by the following geometrical proportion:

As the result of the operation performed on the supposition,
To the true result, or given number,
So is the supposed number
To the number sought.

This rule may be thus elucidated: Let A be the supposed and B the true number, or number sought. Then $\frac{A}{B}$ is the ratio of A to B. But A is operated upon, according to the conditions of the question, by multiplication or division, or both, in the same manner as B, and therefore both are increased or both diminished in the same ratio. Let q represent this ratio; then (167) because $\frac{qA}{qB}=\frac{A}{B}$, it is plain (402) that $qA : qB :: A : B$, and hence the rule.

Examples.

1. A and B trade with equal capital: A gains 25 per cent.; B loses 10 per cent., after which A has \$5000 more than B. What sum had each at first?

Suppose \$10000: Then, as 25 p. c. $=\frac{1}{4}$, and 10 p. c. $=\frac{1}{10}$;
$10000+\frac{10000}{4}=10000+2500=12500$ incr'd capital of A.
$10000-\frac{10000}{10}=10000-1000=\ \ 9000$ dim'd capital of B.
3500 diff. which should be 5000.

Then by the Rule,
$3500 : 5000 :: 10000$, or $7 : 10 :: 10000 : 14285{,}71\frac{3}{7}$, Answer.

Proof.

25 p. c. is $\frac{1}{4}$ | $14285{,}71\frac{3}{7}$
$3571{,}42\frac{6}{7}$
$17857{,}14\frac{2}{7}$ A's incr'd. cap.
$12857{,}14\frac{2}{7}$ B's dim'd. cap.
5000 difference, as by the question.

10 p. c. is $\frac{1}{10}$ | $14285{,}71\frac{3}{7}$
$1428{,}57\frac{1}{7}$
$12857{,}14\frac{2}{7}$

2. A man bequeaths \$5000 to A, B, and C, so that A's part may be double that of B, and the share of B to that of C as 4 to 5. What is the share of each?

Ans. A, \$2352,94$\frac{2}{17}$; B, \$1176,47$\frac{1}{17}$; C, \$1470,58$\frac{14}{17}$.

3. A miller wishes to make a quantity of feed by grinding 400 bush. of barley with peas and beans, so that the peas and beans together may be to the barley as 7 to 3, and the peas double of the beans. How many bushels of each must he grind? Ans. 311$\frac{1}{9}$ bu. beans, and 622$\frac{2}{9}$ bu. peas.

4. Says A to B, My horse without the saddle is worth your horse with my saddle on his back. True, replies B, but my saddle alone is worth 4 times as much as your horse, 3 times as much as your horse and saddle, and, together with $\frac{1}{4}$ the value of my horse, is worth \$1000. What is the value of each horse and saddle?

Ans. A's horse, \$240; B's horse, \$160; A's saddle, \$80; B's, \$960.

Double Position Absolute.

434. Those questions, from the data of which the quantity operated upon must be increased or diminished, not in a certain ratio, require two suppositions, with each of which we find the result as in Single Position. If either of these results accords with the given number, the supposition which gave it must be the number sought. But if both differ, find the difference between each result and the given number, and call this difference an error in plus, when the result is greater; or an error in minus, when the result is less than the given number.

Next write the two suppositions under each other, and opposite each the error which it gave, preceded by its appropriate sign, plus or minus. Multiply the first supposition by the error of the second, and the second supposition by the error of the first, in a cross order. Then, if the errors have like signs, that is, both plus or both minus, *divide the difference of the products by the difference of the errors.* But, if they have unlike signs, that is, the one plus and the other minus, *divide the sum of the products by the sum of the errors.* The quotient is, in either case, the number sought.

Elucidation of the Rule.

Let S' = first supposition, and e' = first error;
S'' = second supposition, and e'' = second errror; and
S = the number sought.

Now when S' is greater than S, it is evident that e' will be plus. Also that the greater S' is, the greater will e' be. *The errors*, therefore, *when both suppositions are greater, are as the excesses of the suppositions;* that is,

$S' - S : S'' - S :: e' : e''$; hence (403)
$S''e' - Se' = S'e'' - Se''$, and by transposition (398)
$Se'' - Se' = S'e'' - S''e'$; or
$S(e'' - e') = S'e'' - S''e'$, and consequetly $S = \dfrac{S'e'' - S''e'}{e'' - e'}$.

It is evident (408) that by inverting the terms of each ratio in the above proportion, or (398) by changing the sign of each term in the equation resulting from it, we shall obtain $S = \dfrac{S''e' - S'e''}{e' - e''}$.

Hence it is of no consequence, as to the rule, which product is the greater.

Again, when S' is less than S, e' will be minus, and the less S' is, the greater will e' be. *The errors*, therefore, when both suppositions are less, *will be to each other* in direct ratio, *as the diminutions of the suppositions;* that is,

$S - S' : S - S'' :: e' : e''$, and hence
$Se'' - S'e'' = Se' - S''e'$, or $Se'' - Se' = S'e'' - S''e'$, or
$S(e'' - e') = S'e'' - S''e'$, or finally $S = \dfrac{S'e'' - S''e'}{e'' - e'}$.

That is to say, when the errors are both plus, or both minus, the difference of the products, divided by the difference of the errors, gives the number sought.

Lastly, for the case in which the signs of the errors are unlike: Let S' be less than S, and consequently e' minus; and let S'' be greater than S, and consequently e'' plus. Then the greater the difference between S' and S, the greater will e' be; also, the greater the difference between S'' and S, the greater will e'' be; therefore, $S - S' : S'' - S :: e' : e''$; and putting this in equation,

$Se'' - S'e'' = S''e' - Se'$, or $Se'' + Se' = S''e' + S'e''$,
or $S(e'' + e') = S''e' + S'e''$, or finally $S = \dfrac{S''e' + S'e''}{e'' + e'}$;

that is to say, when one error is plus and the other minus, the sum of the products, divided by the sum of the errors, gives the number sought.

Double Position Approximate.

435. We must here inform the student that the above is no longer true when the data of the question require the involution of S, or any part of it, into itself; for, in this case, instead of finding, from the formula, the true result, we only arrive at an approximation. Put

$$S' - S : S'' - S : : e' : e''$$

Now suppose the numbers S', S'', and S to have been involved, so that instead of each number, we had its square. Then, that the proportion may remain, it will be necessary to show that the differences of the squares of the numbers are as the differences of the numbers themselves. Let $S' = 8$; $S'' = 10$, and $S = 6$. Then the above proportion becomes $8 - 6 : 10 - 6 : : e' : e''$, or $2 : 4 : : e' : e''$. Then instead of the numbers, taking their squares, if the proportion still exists, we shall have $64 - 36 : 100 - 36 : : 2 : 4$; that is, $28 : 64 : : 2 : 4$, or (403) $112 = 128$, which is absurd.

We shall, however, obtain an approximation. For, let a and $a + q$ be the suppositions, both greater than c, the true number. Also put $a + q = x$; then $a - c : x - c : : e' : e''$, and $\frac{a - c}{x - c} =$ ratio of differences. Then taking the square of each supposition and the square of the true number, we have $\frac{a^2 - c^2}{x^2 - c^2} =$ ratio of the involved numbers. To show the approximation of these ratios, we divide one by the other, thus:

$$\frac{a - c}{x - c} \div \frac{a^2 - c^2}{x^2 - c^2} = \frac{a - c}{x - c} \times \frac{x^2 - c^2}{a^2 - c^2} = \frac{x + c}{a + c} = \frac{a + c + q}{a + c} = 1 + \frac{q}{a + c}.$$

Now, if we take each supposition as near as possible to the true number, their difference q is a very small number, and consequently, $1 + \frac{q}{a + c}$ a near approximation to a unit, which is the result when the ratios are equal.

To solve a question of this kind, therefore, assume two numbers, each as near as possible to the true, and find the result as above, by the appropriate formula. Take the result which is an approximation, and of the two suppositions, that which is nearest to the true number, or any other which may appear still nearer, and proceed as before. The true number may thus, by repeated trials, be obtained to any degree of exactness.

Examples.

1. Two merchants begin business with equal sums of money; one gains 25 per cent., the other loses 10 per cent. of his capital, and $2000 more; after which the first has $5000 more than the other. With what sum did each begin?

25 p. c. is $\frac{1}{4}$, and 10 p. c. is $\frac{1}{10}$.
First suppose 10000; then

$$\begin{array}{rrl} 10000 + 2500 = & 12500 & \text{cap. of first,} \\ 10000 - 1000 - 2000 = & \underline{7000} & \text{" second.} \\ \text{diff.} & 5500 & \text{which should be 5000} \\ & \underline{5000} & \\ & 500 & \text{error in plus.} \end{array}$$

Suppose 8000; then

$$\begin{array}{rrl} 8000 + 2000 = & 10000 & \text{cap. of first.} \\ 8000 - 800 - 2000 = & \underline{5200} & \text{" second.} \\ & 4800 & \text{diff. which should be 5000} \\ & \underline{5000} & \\ & 200 & \text{error in minus; then by the} \end{array}$$

formula,

$$\begin{array}{lrcr} & 10000 \times 2\,00 & = & 20000\,00 \\ & 8000 \times \underline{5\,00} & = & \underline{40000\,00} \\ \text{Sum of errors} & 7{,}00 &) & 60000{,}00 \\ & & & \$8571{,}42\tfrac{6}{7} \text{ required sum.} \end{array}$$

$$\begin{array}{lrl|r} \text{Proof.} & \$8571{,}42\tfrac{6}{7} & & \$8571{,}42\tfrac{6}{7} \\ & \underline{2142{,}85\tfrac{5}{7}} & & \underline{857{,}14\tfrac{2}{7}} \\ & 10714{,}28\tfrac{4}{7} & \text{cap. of first} & 7714{,}28\tfrac{4}{7} \\ & \underline{5714{,}28\tfrac{4}{7}} & \text{" second.} & \underline{2000{,}00} \\ \text{diff.} & \$5000, & \text{as required by the question.} & 5714{,}28\tfrac{4}{7} \end{array}$$

By a Simple Algebraic Equation.

Let $x =$ sum: then $x + \frac{x}{4} =$ capital of first; and $x - \frac{x}{10} - 2000 =$ capital of second; and, by the question, as the difference is 5000, if we add this to capital of second, we have

the equation $x + \frac{x}{4} = x - \frac{x}{10} + 3000$; multiplying by 40, we have $40x + 10x = 40x - 4x + 120000$; subtract $40x$,
and $10x = -4x + 120000$; transpose $4x$,
$14x = 120000$, or $7x = 60000$, and
$x = 60000 \div 7 = \$8571,42\frac{6}{7}$. Answer.

All questions in Double Position, which, in their operation, require neither Involution nor Evolution, belong to Double Position Absolute; and may, as above, be solved, either by one operation, according to the formula, or by a simple equation.

2. What number exceeds five times its square root by 13?

Take 64: then $\sqrt{64} = 8$, and $5 \times 8 = 40$; $64 - 40 = 24$; $24 - 13 = 11$, error in plus.

Take 49: then $\sqrt{49} = 7$; $5 \times 7 = 35$; $49 - 35 = 14$; $14 - 13 = 1$, error in plus.

Mult. in a cross order $64 \times 1 = 64$ and $49 \times 11 = 539$. Then, as the errors have like signs $539 - 64 = 475$, difference of products.

$11 - 1 = 10$ difference of errors, and $475 \div 10 = 47,5$, approximate number.

$\sqrt{47,5} = 6,892$; $5 \times 6,892 = 34,46$
$47,5 - 34,46 = 13,04$; $13,04 - 13 = ,04$, error in plus.

Assuming 49 and 47,5, and multiplying as before, we have
$49 \times ,04 = 1,96$; $47,5 \times 1 = 47,5$; $47,5 - 1,96 = 45,54$, difference of products.

$1 - ,04 = ,96$, difference of errors; then
$45,54 \div ,96 = 47,4375$, a greater approximation, which exceeds 5 times its square root 6,887488 by 13,00006.

Now take 47,4375 error in plus, 00006
47,5 error in plus, 04.

$$\begin{array}{rl} 47,4375 \times ,04 = & 1,8975 \\ 47,5 \times ,00006 = & \underline{,00285} \\ & 1,89465 \text{ difference of products.} \end{array}$$

$,04 - ,00006 = ,03994$ difference of errors.
$1,89465 \div ,03994 = 47,4374$ required number.

$$\begin{array}{rl} 5 \times \sqrt{47,4374} = 5 \times 6,88748 = & \underline{34,4374} \\ & 13 \text{ required difference.} \end{array}$$

By a Quadratic Equation.

The square of $a + b$ is $a^2 + 2ab + b^2$, or $a^2 + 2ba + b^2$. Take away b^2; let $a^2 + 2ba = 21$, and let $b = 2$: then

$a^2+4a=21$. This equation, in which we find *the square of the unknown quantity in the first term, and the quantity itself in the second*, is properly called *quadratic* from the Latin *quadrans*, a square.

In the equation $a^2+2ba=21$, the coefficient of a in the second term is $2b$. Now b is half this coefficient; and, as the square of this half is the part which is wanting to make the first member a complete square, we infer that in every equation of this kind, if *we add the square of half the coefficient of the second term to each member*, that member which contains the unknown quantity becomes a complete square; and consequently, *by extracting the square root of each side*, the index is expunged, and the equation becomes simple.

In the equation $a^2+4a=21$, the coefficient of a being 4, we add the square of its half to each side, and have

$$a^2+4a+4=25.$$

Now $a+b$, which is the root of $a^2+2ab+b^2$, is merely the sum of the roots of the first and last terms of the expression. Also, as the square of $a-b$ is $a^2-2ab+b^2$, we may infer that the root of the complete square is *the root of the first term, and the root of the last connected by the sign preceding the second term.*

Hence, by extracting the square root of each side of the equation $a^2+4a+4=25$, we have $a+2=5$, or $a=5-2=3$; and substituting 3 for a in the equation $a^2+4a=21$, we have $9+12=21$.

To solve example 2, let $x=$ number required. Then $x-13=5\sqrt{x}$; divide by 5, $\frac{x-13}{5}=\sqrt{x}$; square both sides, $\frac{x^2-26x+169}{25}=x$; mult. by 25, $x^2-26x+169=25x$; transpose $25x$ and 169, $x^2-51x=-169$; to complete the square, we add $\frac{51}{2}\times\frac{51}{2}=\frac{2601}{4}$ to each side, and have

$x^2-51x+\frac{2601}{4}=\frac{2601-676}{4}=\frac{1925}{4}$; extracting roots,

$x-\frac{51}{2}=\sqrt{\frac{1925}{4}}$, $x=\frac{51}{2}+\frac{\sqrt{1925}}{2}=\frac{51+\sqrt{1925}}{2}$, or

$x=\frac{51+43{,}8748}{2}=\frac{94{,}8748}{2}=47{,}4374$ required number.

From the above it is evident that questions in this rule, in which the required number depends on the increase or diminution of the given number by quantities obtained by its involution or evolution, can only be solved by Approximate Double Position or by a Quadratic Equation.

3. A father gives, in unequal portions, $10000 to his two sons. If $1000 be taken from the portion of the younger, the remainder will be to the portion of the elder as 3 to 5. What is the cube root of the ratio of the whole portion of the younger to that of the elder? Ans. ,9196413 +.

4. Having diminished a number by four times its square root, the remainder, diminished by 4, is $\frac{2}{3}$ of the number. What is the ratio of the last remainder to $\frac{1}{3}$ of the square root of the number? Ans. 1.

5. Required a number, from which, if you take 5 times its square root, the remainder will be 3.

Ans. 30,7069 + approx. number.

Alligation.

436. Alligation, which means tying together, is the rule which determines the value of compounds formed of ingredients of different values, and is of two kinds, Medial and Alternate.

Alligation Medial.

This rule is so called, because when several quantities, and the respective value, rate, or price of each is given, it finds the medium, or mean value, rate, or price of the mixture compounded of those quantities. The rule is as follows:

Multiply each quantity by the value of its principal unit, and add the products together: the sum is the value of the whole composition. Next find the sum of the given quantities; then as the mixture, when compounded, is a uniform quantity, it is plain that the values of any two quantities of the mixture must be to each other as the quantities themselves; we therefore say

As the sum of the given quantities
To any proposed part of the mixture,
So is the value of the whole
To the value of the part proposed.

The price or value of a number of units is evidently the result of the price of one unit, added to itself as often as the unit is added in forming the number; that is, the price of one unit multiplied by the number of units: hence, inversely, the price of the whole divided by the number of units, gives the price of one unit.*

Therefore, the value of the whole composition, divided by the number of units which enter into it, will give the value of a unit of the mixture. Also, the value of any number of units may easily be found by multiplying the value of one unit by that number.

Examples.

1. A miller mixes for grinding 60 bushels of corn at 62½ cts.; 80 at 45 cts.; 100 at 40 cts.; with 120 bu. of oats at 37½ cts. a bushel. What is the mixture worth per bushel?

60	@ 62½ cts.	$37,50
80	" 45	86,00
100	" 40	40,00
120	" 37½	45,00
36|0		6)15|8,50
		6)2,6 41 +
		,4 40 +

Hence, the worth of a bushel is 44 cents, very nearly. For 360 × 44 = 158,40, which is less than the true number by only $\frac{1}{1585}$ of that number.

2. Suppose that, during the ninth month, the thermometer stands for 4 days at 66°, for 12 days at 62°, for 8 days at 72°, and for 6 days at 56°. What is the mean temperature?

Ans. 64 degrees.

3. A vintner mixes 20 gals. wine at 6 s. 8 d. per gal., 35 at 6 s., 60 at 4 s. 3 d., and 120 at 3 s. 6 d. Now he offers to sell me 47 gals. at prime cost: how much must I pay, and at what price must he sell the remainder so as to gain 8 d. per gal. on the whole?

Ans. I pay £10 3 s. 8 d., and he sells at 6 s. 3 d. per gal.

Alligation Alternate.

437. Alligation is called Alternate when the prices or values of several ingredients are given, to determine what quantity of each must be taken, to make a mixture of a fixed medium value; *because the given values are tied or linked to-*

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* It must be observed that though the quantities differ from each other, they must all be measured by a unit of the same magnitude; as, for example, bushels of barley must be compounded with bushels of oats, corn, &c.; pounds of mustard with pounds of meal, &c. &c.

gether, two and two, so that one may be greater, and the other less, than the medium rate, alternately.

Let m be the medium rate. Let A be a greater and B a less rate than m. Also, let $A = m + r$, and $B = m - s$. Then, if P be any number of units taken at the value A, that is, at $m + r$, their value will be $Pm + Pr$, that is, Pr too great; and if q be a number taken at the value B or $m - s$, their value will be $qm - qs$, that is, qs too little. But, in order that the composition, formed of the two quantities, may be at the mean rate, Pr *must equal* qs; therefore, (406) $P : q :: s : r$; that is to say, *the quantities, taken at the respective values, are reciprocally proportional to the differences between those values and the mean rate.* The ratio of the required quantities being thus determined by the differences between their values and the mean rate, we assume those differences as the required numbers, and proceed according to the following rule:

Place the difference between the greater value and the mean rate opposite to the less value, with which it is linked; and the difference between the less value and the mean rate opposite to the greater value.. Do this with each couple which is linked. Then, when more than one number stands opposite to any value, which will often happen,—and, indeed, must always happen when the number of ingredients is odd,—their sum is the required number at that rate. Multiply each value by the number opposite to it, and add the results, the sum of which will be the value of the whole mixture. Divide this sum by the whole number of units in the mixture, the quotient will be the mean rate, when the work is right; and is, therefore, a proof of the operation.

Examples.

1. A vintner has wines at 3 s., 4 s. 6 d., and 7 s. 6 d. a gallon, which he would mix so that the compound may be worth 5 s. a gallon. How many gallons of each may he take?

5 s.	3 s. 0 d.	2 s. 6 d. or	2½	7 s.	6 d.
	4 s. 6 d.	1 s. 0 d.	1	4	6
	6 s. 0 d.	6 d.	½	3	0
	7 s. 6 d.	2 s.	2	15	0

Number of gals. 6) 30 s. whole cost of comp'd.

5 s. required rate per gal.

Here we see, that by mixing 2½ gals. at 3 s., 1 gal. at 4 s.

6 d., ½ gal. at 6 s., and 2 gals. at 7 s. 6 d., the whole composition is worth 30 s., which, divided by 6, the whole number of gallons, gives 5 s. for the average or mean price per gallon, as required.

Because 3 s. × 2½ = (5 — 2) 2½ is 2 × 2½ too little; and (7 s. 6 d.) 2, or 7½ s. × 2 = (5 + 2½) 2 is 2 × 2½ too great: the sum of the products must be equal to 2 + 2½, or 4½, at the mean rate.

Different results may be obtained *either by a variation or complication of the linking.*

$$5\text{ s.}\left\{\begin{array}{rr|rrr} 3\text{ s.} & 0\text{ d.} & 1\text{ s.} & 3\text{ s.} & \\ 4 & 6 & 2\tfrac{1}{2} & 11 & 3 \\ 6 & 0 & 2 & 12 & 0 \\ 7 & 6 & \tfrac{1}{2} & 3 & 9 \\ \hline & & 6\,)\, & 30\text{ s.} & \\ & & & 5\text{ s.} & \end{array}\right.$$

or,

$$5\text{ s.}\left\{\begin{array}{rr|lllll} 3\text{ s.} & 0\text{ d.} & 2\tfrac{1}{2}+1 & = 3\tfrac{1}{2} & — 10\text{ s.} & 6\text{ d.} \\ 4 & 6 & 1+2\tfrac{1}{2} & = 3\tfrac{1}{2} & — 15 & 9 \\ 6 & 0 & \tfrac{1}{2}+2 & = 2\tfrac{1}{2} & — 15 & 0 \\ 7 & 6 & 2+\tfrac{1}{2} & = 2\tfrac{1}{2} & — 18 & 9 \\ \hline & & & 12\ ) & 60\text{ s.} & \\ & & & & 5\text{ s.} & \end{array}\right.$$

When one number is multiplied by another, the product (138) is the same as the sum of the products when all its parts are separately multiplied by that other; therefore, if we multiply the numbers 2½, 1, ½, and 2 of the first linking, each by 12, the sum of the products will be 12 times their sum 6. Also, as each of the products 7 s. 6 d., 4 s. 6 d., 3 s., and 15 s. will be multiplied by 12, their sum 30 s. will be multiplied by 12. But (165) because the quotient is not altered when the divisor and dividend are both multiplied or both divided by the same number, we shall obtain the same result. Hence it follows that if, after having found a series of numbers that complies with the conditions of the question, we increase or diminish the numbers it contains, each in the same ratio, whether that ratio is integral, fractional, or mixed, we shall still obtain the same result; consequently, *we can obtain new series without end; find numbers corresponding to a given or limited number of one or more of the ingredients, or, finally, limit the whole compound to a certain quantity.*

438. Suppose, for example, *one ingredient being limited,* that in the above compound we would have 25 gals. at 3 s., we say,
$2\frac{1}{2} : 25 :: 1 : 10$ gals. @ 4 s. 6 d.
$2\frac{1}{2} : 25 :: \frac{1}{2} : 5$ gals. @ 6 s.
$2\frac{1}{2} : 25 :: 2 : 20$ gals. @ 7 s. 6 d. That is to say,
As the number in the series which stands opposite to the rate of the given number is to the given number, so is each successive number in the series to the quantity required at its rate.

As the ratio of 25 to $2\frac{1}{2}$ is 10, the numbers corresponding to 25 might easily be found by multiplying each remaining number in the series by 10.

$$\begin{array}{lllr}
\text{Proof.} & 25 & @\ 3\text{ s.} & = 75 \\
 & 10 & @\ 4\text{ s. }6\text{ d.} & = 45 \\
 & 5 & @\ 6\text{ s.} & = 30 \\
 & 20 & @\ 7\text{ s. }6\text{ d.} & = 150 \\
\hline
 & 6|0 & &)\ 30|0 \\
\cline{4-4}
 & & & 5\text{ s.}
\end{array}$$

439. Again, *two or more ingredients being limited,* suppose we would have 30 gals. at 3 s., and 24 at 4 s. 6 d. *We first find, by Alligation Medial, the mean rate for the given ingredients.*

$$\begin{array}{lcr}
30 \times 3 & = & 90 \\
24 \times 4\frac{1}{2} & = & 108 \\
\hline
54 &) & 198\ (3\frac{36}{54} = 3\frac{2}{3}\text{ s., or 3 s. 8 d. mean rate.} \\
 & & 162 \\
\cline{3-3}
 & & 36
\end{array}$$

Then, reducing all the rates to pence, *we find a series for this mean rate, and the rates of the undetermined ingredients.*

$$60\left\{\begin{array}{l}44\\72\\90\end{array}\right. \quad \begin{array}{rl}30 + 12 = 42 & - 1848 \\ 16 & - 1152 \\ 16 & - 1440 \\ \hline 74\ ) & 4440\ (60 \\ & 444 \\ \cline{2-2} & *** \end{array}$$

Lastly, we say: *As the sum of the numbers opposite to the mean rate of the given ingredients to the sum of those ingredients, so is the number opposite to any other rate to the number required at that rate.*

Hence we have $42 : 54 :: 16$, or $7 : 9 :: 16 : \frac{144}{7} = 20\frac{4}{7}$

@ 6 s. The remaining number being the same, we shall have $20\frac{4}{7}$ @ 7 s. 6 d.

Proof. 30 @ 3 s. $=$ 90
24 @ 4 s. 6 d. $=$ 108
$20\frac{4}{7}$ @ 6 s. $=$ $123\frac{3}{7}$
$20\frac{4}{7}$ @ 7 s. 6 d. $=$ $154\frac{2}{7}$
$95\frac{1}{7}$) $475\frac{5}{7}$ (5 s.
$475\frac{5}{7}$
***

440. *The whole compound being limited,* suppose we would limit the above compound to 100 gallons. Take any of the above series, or *first find a series, at pleasure, for the given rates; then say, As the sum of the series found is to the given number* 100, *so is any number in the series to the corresponding part of the given or limited number* 100.

Taking the series 25, 10, 5, and 20, we say $25 + 10 + 5 + 20 = 60$. Then, as the ratio $60 : 100 = 3 : 5$, we have

3 : 5 : : 25 : $\frac{125}{3}$ $=$ $41\frac{2}{3}$ gals. @ 3 s.
3 : 5 : : 10 : $\frac{50}{3}$ $=$ $16\frac{2}{3}$ gals. @ 4 s. 6 d.
3 : 5 : : 5 : $\frac{25}{3}$ $=$ $8\frac{1}{3}$ gals. @ 6 s.
3 : 5 : : 20 : $\frac{100}{3}$ $=$ $33\frac{1}{3}$ gals. @ 7 s. 6 d.

Proof. $41\frac{2}{3}$ @ 3 s. $=$ 125
$16\frac{2}{3}$ @ 4 s. 6 d. $=$ 75
$8\frac{1}{3}$ @ 6 s. $=$ 50
$33\frac{1}{3}$ @ 7 s. 6 d. $=$ 250
100) 500 (5 s. required rate.

2. A grocer has sugar at 3 d., 4 d., 8 d., and 10 d. per lb. How many pounds of each must he mix with 60 lbs. at 5 d. so that the whole may be worth 6 d. per lb.?

Ans. 60 at 3 d., 60 at 10 d., 120 at 4 d., and 120 at 8 d.

3. A dealer in spirits mixes 60 gals. of brandy at $3 per gal. with 75 gals. at $2,50. How much water must be add to enable him to sell the mixture at $2 per gal.?

Ans. $48\frac{3}{4}$ gals.

4. A miller has 90 bush. corn at $62\frac{1}{2}$ cts., 80 at 50 cts., and 60 bush. at 60 cts., which he would mix with oats at $37\frac{1}{2}$ cts. a bushel, so that the whole may be worth 50 cts. a bushel. How many bushels of oats must he have?

Ans. 138 bushels.

5. A miller has 40 bushels of grain at 60 cts. which he

would mix with grain at 35 cts. and at 56 cts.; how much of each of the two last must he take to form a compound worth 50 cts. a bushel, which shall fill a garner 10 ft. long, 5 ft. $4\frac{1}{2}$ in. wide, and 5 ft. high.

Ans. $69\frac{1}{3}$ bush. at 35 cts., and $106\frac{2}{3}$ at 56 cts.

6. How many bushels of grain at 40, 50, 60, and 80 cts. a bushel must we have to form a compound worth 56 cts. a bushel, which shall just fill the above garner?

Ans. $103\frac{17}{25}$ at 40 cts., $17\frac{7}{25}$ @ 50, $25\frac{23}{25}$ @ 60, and $69\frac{3}{25}$ @ 80 cts.

Permutation.

441. The changes which can be made in the order of any given number of things, the whole being disposed in regular succession after each other, and each being found in every arrangement only once, are called *permutations.*

Suppose we have two things *b* and *c*: it is plain that they are susceptible of the two permutations *bc* and *cb*, which we may express by 1 × 2. If we have three things *a*, *b*, *c*: as any one of them may stand first, each remaining two being susceptible of two permutations, it is plain that we shall have 3 times 2, or 6 permutations, namely, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, which number is expressed by 1 × 2 × 3, or $1 . 2 . 3$.

Again, if we have four things: as any three of the four are susceptible of 6 permutations, and as each may be placed first in its turn, it is evident that the number of permutations of which they are susceptible is 4 times $6 = 24$, or $1 . 2 . 3 . 4$.

Now it is easy to see that if there be 5 things, the number of changes in their order will be 5 times 24, or $1 . 2 . 3 . 4 . 5$; and that in general, whatever is the number of things, the number of permutations of which they are susceptible is expressed by the continual multiplication of the natural numbers $1 . 2 . 3 . 4 . 5$, &c., ending with the number corresponding to the given number of things.

Supplement to Permutation.

442. When, of the things which are given, several are alike, as *a*, *a*, *a*, *b*, it is plain that as those things which are alike are not susceptible of permutation, the number of permutations of which the like things would have been susceptible, had they all differed from each other, shows *how many times less the number of permutations of which the given things are susceptible now is, than it would have been had they been all differ-*

ent. We therefore operate thus: Find the number of permutations of which the given things would have been susceptible had they been all different, as usual; and divide this number by the product of the numbers which show the number of permutations of which all the like things would have been susceptible, if different.

Thus, for the four things a, a, a, b, we have $\frac{1 . 2 . 3 . 4}{1 . 2 . 3} = 4$, the number of permutations of which they are susceptible. These are in effect, *aaab*, *aaba*, *abaa*, *baaa*, and are evidently all of which they are susceptible, seeing that b is put in every place from the first to the last inclusive, and that the letters a, a, a, are not susceptible of permutation.

For the permutations of the ten letters *aabbbbcccd*, we have

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}{(1 . 2)\,(1 . 2 . 3 . 4)\,(1 . 2 . 3)} = 12600.$$

Examples.

1. How many changes can be made in the order of the 5 vowels? Ans. 120.

2. In how many ways can the seven notes of music be played? Ans. 5040.

3. How many different numbers, each consisting of 9 figures, can be expressed with the 9 digits? Ans. 362880.

4. How many numbers, each consisting of ten figures, can be expressed with the figures 7, 1, 2, 3, 8, 3, 2, 9, 2, 6?
Ans. 302400.

5. In how many ways can a vignette be formed, which shall consist of 2 violets, 3 primroses, 4 tulips, and 5 roses?
Ans. 2522520.

Arrangement.

443. When we have a number m of things, and are required to find how many changes can be made in their order when they are taken 2 by 2, 3 by 3, 4 by 4, &c., n by n, n being always assumed less than m, these changes are called arrangements.

Suppose we have 5 things a, b, c, d, e, and we wish to know how many changes can be made in their order by taking them 2 by 2. As any one can precede each of the others, it is evident that 4 arrangements will thus result from each, and consequently that 5 things taken 2 by 2 are susceptible of 5 times 4, or 20 arrangements.

Now if we have any number m of things, it is equally plain that by placing any one before each of the others, the number of arrangements thus formed will be $(m - 1)$, and that by thus placing the whole of the units in m, each in its turn, we shall have m times $(m - 1)$, or $m(m - 1)$ for the whole number of arrangements of which m things are susceptible when taken 2 by 2.

Again, if we take 5 things 3 by 3, we easily see that, before the whole of the arrangements obtained in placing them 2 by 2, we can place each one of the remaining 3, and thus we shall have 3 times 20, or 60 arrangements. Also, when the things in a number m are arranged 2 by 2, we can, before every arrangement, place each of those which remain, the number of which is $(m - 2)$; consequently, as the number of arrangements 2 by 2 is $m(m - 1)$, the number of arrangements 3 by 3 must be $(m - 2)$ times $m\ (m - 1)$; or, inverting the order of the multiplication, $m\ (m - 1)\ (m - 2)$.

Hence we see, that the number of arrangements of which a number m of things is susceptible, when taken 2 by 2, is $m(m - 1)$, and when taken 3 by 3, is $m(m - 1)\ (m - 2)$.

444. Now, if we take the given number of things n by n, n being less than m, it is easy from the above reasoning to infer that the number of arrangements, n by n, of which m things are susceptible, is expressed by the continual multiplication of the decreasing series $m(m - 1)\ (m - 2)\ (m - 3)\ (m - 4)$, &c., in the last term of which *the negative number is equal to* $(n - 1)$; and as this number is to be subtracted from m, the last term in the series will be $m - (n - 1)$, or $(m - n + 1)$.

The general formula, therefore, for finding the number of arrangements of m things, taken n by n, is $m(m - 1)\ (m - 2)\ (m - 3)$, &c......$(m - n + 1)$.*

Examples.

1. Suppose we demand how many different numbers, each consisting of 4 figures, can be formed with the six digits 1, 2, 3, 4, 5, 6.

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* The negative number in the second term being 1, and that in the last $(n - 1)$, it is plain that the number of terms succeeding m is $(n - 1)$; consequently, the whole number of terms will always equal n. Wherefore, to find the number of arrangements, we multiply a series, beginning with the given number of things, and decreasing by a unit, till the number of terms in the series is equal to the number of things taken at a time.

The last term of the series will, according to the formula, be $(6-4+1)=3$; we shall, therefore, have

$$6 \times 5 \times 4 \times 3 = 360, \text{ the required number.}$$

2. How many different whole numbers, each consisting of 5 figures, can be formed with the 9 digits? Ans. 15120.

3. How many different whole numbers, each consisting of 7 figures, can be expressed with the nine digits? Ans. 181440.

4. If from a regiment consisting of 10 companies, we take 6 companies at a time, in how many ways can the 6 companies take up their order of march? Ans. 151200.

Combination.

445. Of the arrangements of which m things are susceptible when taken 2 by 2, 3 by 3, &c., n by n, n being less than m, those which differ from each other by at least one of the things of which they are composed, are called *combinations*. Thus, the letters a, b, c, taken 2 by 2, are susceptible of the 6 arrangements ab, ac, ba, bc, ca, cb, of which only 3, ab, ac, and bc, are combinations.

Now, as the combinations of m letters, taken n by n, all differ from each other by at least one of the letters in n, it is plain that all the permutations of which the n letters of each combination are susceptible are so many new arrangements of the m letters taken n by n, and comprise all the arrangements of which the m letters, so taken, are susceptible. Hence the whole number of arrangements, of which m letters, taken n by n, are susceptible, is the product of the combinations of the same m letters, taken n by n, multiplied by the number of permutations of which n letters are susceptible. Wherefore, calling the whole number of arrangements of the m letters A, the combinations C, and the permutations of n letters P, it is plain, that $A = PC$, and consequently $C = \frac{A}{P}$. That is to say, the number of combinations of m letters, taken n by n, is equal to the whole number of their arrangements when so taken, divided by the number of permutations of which the n letters are susceptible. But (444) $A = m(m-1)(m-2)(m-3)$ &c......$(m-n+1)$ and (441) $P = 1 . 2 . 3 . 4 . 5$, &c..... $\times (n-1) \times n$. Wherefore, the expression $C = \frac{A}{P}$ becomes $C = \frac{m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)}{1 \cdot 2 \cdot 3 \cdot 4, \&c.\ldots\ldots \times (n-1)n}$.

Examples.

1. Suppose that having 12 different kinds of flowers, we would know the whole number of nosegays, each consisting of 7 different kinds, that we can with the 12 kinds form, so that no two nosegays may consist of exactly the same kinds of flowers.

It is evident that, to solve this question, we have only to find the combinations of 12 things taken 7 by 7. Wherefore, by the above formula, we have $\frac{12\times11\times10\times9\times8\times7\times6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$ $= 792$, the required number.

2. With 9 different kinds of fruit, how many persons can we regale, so that each may partake of 4 kinds, and no two persons exactly the same kinds? Ans. 126.

3. How many different numbers can we form by multiplying 6 at a time of the first 12 prime numbers? Ans. 924.

4. From 40 men, how many times could a commander select 20, without again choosing the same 20?
Ans. 137846528820 times.

Supplement to Combination.

446. If we have two sets of series of different letters, which we would combine 2 and 2, it is evident that in combining a letter of the first set with each of those in the second, we shall have as many combinations, 2 and 2, as there are letters in the second set; wherefore, in thus combining every letter of the first set, it is plain that all the combinations, 2 and 2, of which the two sets are capable, in choosing one from each, is signified by their product. Again, if we have a third set, and we would combine the whole 3 by 3, by taking one from each, it is plain that in combining each letter of the third set with each of the combinations, 2 and 2, of the two first, we shall have the number of combinations, 3 and 3, which is signified by the product of the first and second series. Wherefore, the whole number of combinations of which 3 sets are capable, when taken 3 by 3, is signified by the product of the three sets; and, pursuing the analogy, all the combinations which can be made from any number of sets of different things, by taking, to form each combination, one from each set, is expressed *by the continual multiplication of the numbers which show the number of things in each set.*

Examples.

1. From a company of 6 Germans, 10 Frenchmen, 7 Russians, 8 Americans, and 9 Spaniards, in how many ways can 5 persons be selected, so as to have one of each nation?

Ans. 30240.

2. With 12 roses, 7 tulips, 13 carnations, 6 anemones, 5 ranunculuses, and 10 wall-flowers, in how many ways can a bouquet be formed, consisting of one flower of each kind.

Ans. 327600.

SECTION XXIII.

NEWTON'S BINOMIAL FORMULA—ARITHMETICAL PROPORTION AND PROGRESSION—GEOMETRICAL PROGRESSION—LOGARITHMS—USE OF THE TABLE.

Newton's Binomial Formula.

447. We have shown (444) that in the series $m(m-1)$, $(m-2)$, &c., for finding the number of arrangements of m things, taken n at a time, the number of terms is always equal to n. The number of combinations, therefore, of m things, taken 1 by 1, is m, or $\frac{m}{1}$; taken 2 by 2, it is $\frac{m(m-1)}{1\,.\,2}$; taken, 3 by 3, it is $\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}$, &c. Now when m is an even number, it is divisible by 2; also, when m is odd, $(m-1)$ is divisible by 2; consequently, $\frac{m(m-1)}{1\,.\,2}$ is a whole number. Again, when m is not divisible by 3, the remainder found in dividing it by 3 must be either 1 or 2. Now, when the remainder is 1, $(m-1)$ is divisible by 3; and, when the remainder is 2, $(m-2)$ is divisible by 3; wherefore, the expression $m(m-1)(m-2)$ is divisible by 3, and we have shown that it is divisible by 2; therefore, $\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}$ is a whole number. Wherefore, in general, as the remainder, in dividing any number m by n, must be some one of the numbers 1, 2, 3, &c. not exceeding $n-1$; and as m is successively diminished by each of those num-

bers to $n-1$, inclusive, it is plain that the number n exactly measures some one of the numbers in the scale $m(m-1)$, $(m-2)$, $(m-3)$, &c. Hence, if we make, successively, each number in the scale 1, 2, 3, 4, &c. equal to n, we shall show, as above, that the formula $\frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4}$, &c., consisting of any number of terms, is a whole number.

This formula the student will recognize as the formula for finding the numeral coefficients in the Binomial Theorem of Newton.

Demonstration of the Binomial Formula.

448. Let m be the index of the power to which a binomial is raised; the coefficients of any term which is preceded by n terms is the number of combinations of m things, taken n by n. This law or property M. Bourdon, in his Elémens D'Algèbre, has shown as follows:

"The more easily to discover the law of the developement of the m^{th} power of the binomial $x+a$, we shall begin by observing the law which exists in the product of several binomials $x+a$, $x+b$, $x+c$, $x+d$ . . . having a common first term, and their second terms different. The object of this artifice is to prevent the reduction of the similar terms.

$$\begin{array}{l} x+a \\ \underline{x+b} \end{array}$$

1st product, $x^2 \left.\begin{array}{l}+a\\+b\end{array}\right\} x+ab.$

$$\underline{x\ +c}$$

2d product, $x^3 \left.\begin{array}{l}+a\\+b\\+c\end{array}\right\} x^2 \left.\begin{array}{l}+ab\\+ac\\+bc\end{array}\right\} x+abc.$

$$\underline{x\ +d}$$

$$x^4 \left.\begin{array}{l}+a\\+b\\+c\\+d\end{array}\right\} x^3 \left.\begin{array}{l}+ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{array}\right\} x^2 \left.\begin{array}{l}+abc\\+abd\\+acd\\+bcd\end{array}\right\} x+abcd.$$

"Having performed the multiplications according to the usual rules, we discover in the three preceding products the following law:

"1. With regard to the exponents or indices, the index of x

is, in the first place, equal to the number of binomials multiplied. This index then diminishes, by a unit at a time, in each successive term, to the last, where it is equal to nought.

"2. With respect to the coefficients of the different powers of x, the coefficient of the first term is a unit; the coefficient of the second term is equal to the sum of the second terms of the binomials; the coefficient of the third term is equal to the sum of the different products, or combinations of the same second terms taken 2 by 2; the coefficient of the fourth term is equal to the sum of the different products or combinations, 3 by 3. Pursuing the analogy, we may say that the coefficient of any term, which has n terms before it, is equal to the number of different products or combinations of the second terms of the binomials, taken n by n."

Now the number of second terms is equal to the number of binomials, or index of the power of their leading quantity x. Hence, we infer that, when a binomial is raised to any power m, the index of the power, or m, is the coefficient of the second term. The coefficient of the third term is the number of combinations of the same number m, taken 2 and 2. But this is $\frac{m(m-1)}{1\ .\ 2}$; that is, it is the coefficient of the second term, multiplied by the index of the leading quantity in that term, and divided by the number which shows the number of terms to that place, or number preceding the third term. The coefficient of the fourth term being the combinations of the same number m, taken 3 by 3, is $\frac{m(m-1)(m-2)}{1\ .\ 2\ .\ 3}$; that is, it is the coefficient of the third term, multiplied by the index of the leading quantity in that term, and divided by the number showing the number of terms preceding the fourth. Wherefore, in general, the coefficient of any term is found by multiplying the coefficient of the term preceding it, by the index of the leading quantity in that term, and dividing the product by the number which shows the number of terms preceding the term the coefficient of which is sought.

449. The number of combinations of which m things are susceptible is the same when the m things are taken 1 by 1, as when taken $(m-1)$ by $(m-1)$; when they are taken 2 by 2, as when they are taken $(m-2)$ by $(m-2)$; and, in general, when taken n by n, as when taken $(m-n)$ by $(m-n)$, n being less than m.

Let $n = 1$; then $(m - n + 1) = m$; that is, m is the number of combinations, when taken 1 at a time.

Let $n = m - 1$; then $(m - n + 1) = m - (m - 1) + 1$, or $m - m + 1 + 1 = 2$. Wherefore, the formula

$$\frac{m(m-1)(m-2)(m-3)\ \ldots\ 2}{1\ .\ 2\ .\ (n-3)(n-2)(n-1)n},$$

becomes $\dfrac{m\,|\,(n)(n-1)(n-2)(n-3),\ \&c.\ 2}{1\ .\ 2\ .\ (n-3)(n-2)(n-1)\qquad n} = \dfrac{m}{1} = m.$ That is to say, when m things are taken $(m - 1)$ at a time, the number of combinations is equal to m, the same as when taken 1 at a time.

Let $n = 2$; then $(m - n + 1) = m - 1$, and the number of combinations is $\dfrac{m(m-1)}{1\ .\ 2}$.

Again, let $n = m - 2$; then $(m - n + 1) = m - (m - 2) + 1 = m - m + 2 + 1 = 3$. Wherefore, the formula $\dfrac{m(m-1)(m-2)(m-3)(m-4)\ \ldots\ 3}{1\ .\ 2\ .\ 3\quad (n-2)(n-1)\qquad n}$ becomes

$$\frac{m(m-1)\,|\,(n)(n-1)(n-2)\ldots 3}{1\ .\ 2\ .\,|\,3\ (n-2)(n-1)\qquad n} = \frac{m(m-1)}{1\ .\ 2};$$

which shows that the number of combinations is the same when m things are taken 2 at a time as when taken $m - 2$ at a time. Hence, we may infer that the number of combinations is the same when m things are taken n at a time as when $(m - n)$ at a time.

450. Also, the number of combinations is greatest when $n = \dfrac{m}{2}$; that is, when n is the half of m.

For, as the last term in the numerator of the fractional formula is $(m - n + 1)$, and the last term in the denominator is n, it is plain that as long as $(m - n + 1)$ is greater than n, the greater n is; that is to say, (444,) the greater the number of terms, the greater will be the number of combinations; but, when $(m - n + 1)$ becomes less than n, the greater the number of terms, the less must be the result, or number of combinations, because the series which constitutes the numerator of the formula is decreasing, while that of the denominator is increasing. Now it is plain that $(m - n + 1)$ will be greater than n, when $n = \dfrac{m}{2}$; but when $n = \dfrac{m}{2} + 1$, that is, when n is a unit greater than the half of m, then

$$m - n + 1 = m - \frac{m}{2} - 1 + 1 = m - \frac{m}{2} = \frac{2m - m}{2} = \frac{m}{2},$$

and is, therefore, less than n. Wherefore, the number of combinations is greatest when n is the half of m.

451. When m is odd, and consequently its half fractional, n, being integral, cannot equal $\frac{m}{2}$; but, taking n as near $\frac{m}{2}$ as possible, $m - n$ must equal $n + 1$, or $n - 1$. First, if $m - n = n + 1$, then $m - n + 1 = n + 2$. Again, if $m - n = n - 1$, then $m - n + 1 = n - 1 + 1$, or n; and as these are both equally near $\frac{m}{2}$, the number of combinations must be, in either case, the same.

For example, let us take m equal to the odd number 9; then $\frac{m}{2} = \frac{9}{2} = 4\frac{1}{2}$. Now, taking n as near as we can, in integers, to $4\frac{1}{2}$, n will be 4 or 5; and, as these are equally near, the number of combinations must, in either case, be the same. First, let $n = 4$; then $m - n + 1 = 6$, and, by the formula, we have $\frac{9 \times 8 \times 7 \times 6}{1 \,.\, 2 \,.\, 3 \,.\, 4} = 189$, for the number of combinations. Again, taking $n = 5$, we have $m - n + 1 = 5$, and $\frac{9 \times 8 \times 7 \times 6 \times | 5}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, | 5} = 189$, as before.

Now, applying this to the coefficients of the binomial, we may observe that when the power is odd, the number of terms is even, and we always have two middle terms, which are alike, and the greatest in the scale. But when the power is even, and, consequently, the number of terms odd, we have only one middle term, the coefficient of which is the greatest.

Arithmetical Proportion.

452. The difference between two quantities is called their *arithmetical ratio.* Thus 3 is the arithmetical ratio of 5 to 2, or of 7 to 4. An arithmetical ratio is written thus: 5 . 2, and is read 5 is to 2. An arithmetical proportion is formed of two equal arithmetical ratios.

When four quantities a, b, c, d are such that $a - b = c - d$, or $b - a = d - c$, they, and only such, are said to be in arithmetical proportion. The proportion is written thus: $a \,.\, b : c \,.\, d$, and is read a is to b, as c is to d.

In an arithmetical proportion, *the sum of the extremes is equal to the sum of the meuns*: and this, which is called the *fundamental property*, may be thus demonstrated: Let $a \centerdot b : c \centerdot d$ be a general representation of every arithmetical proportion. First, let the antecedents, a and c, be greater than their consequents b and d; and let $a - b = q$, then $c - d = q$, because $a - b = c - d$. Hence, $a = b + q$, and $c = q + d$; consequently, $a + d = b + q + d$, and $b + c = b + q + d$; wherefore, $a + d = b + c$.

Again, let the consequents b and d be greater than their antecedents, and let $b - a = q$; then $d - c = q$. Also, $b = a + q$, and $d = q + c$; and hence, $a + d = a + q + c$, and $b + c = a + q + c$; consequently, $a + d = b + c$, as before. Thus, in the proportion $5 \centerdot 2 : 7 \centerdot 4$, we have $5 + 4 = 7 + 2$. Wherefore, in every arithmetical proportion *the sum of the extremes is equal to the sum of the means.*

453. From this fundamental property, it is evident, that we may change the places of the means or those of the extremes; also, that we may take the means for the extremes, and the extremes for the means, without disturbing the proportion, because, in all these changes, the sum of the extremes will be equal to that of the means.

454. When in an arithmetical proportion, the two middle terms are alike, the proportion is said to be *continual*. Thus, $2 \centerdot 5 : 5 \centerdot 8$, or $a \centerdot b : b \centerdot c$, is a continual proportion; and is sometimes written without repeating the middle term, thus: $\div 2 \centerdot 5 \centerdot 8$, or $\div a \centerdot b \centerdot c$. The preceding sign $\div$ is used, not only to distinguish the proportion from a common multiplication, but to signify the repetition of the middle term.

The middle term is called a *mean proportional*, or *arithmetical mean*, between the other two. In the above proportions, 5 is an arithmetical mean between 2 and 8; and b, an arithmetical mean between a and c.

455. As the sum of the extremes is equal to that of the means, it is evident that we shall find the arithmetical mean between any two numbers or quantities *by taking half their sum*. Thus, to find a mean between 5 and 8, we have $\frac{5+8}{2} = \frac{13}{2} = 6\frac{1}{2}$; therefore, $\div 5 \centerdot 6\frac{1}{2} \centerdot 8$, or $\div 8 \centerdot 6\frac{1}{2} \centerdot 5$.

The difference which reigns in the proportion between term and term, is called *the ratio*.

Arithmetical Progression.

456. This name is given *to a series of numbers in continual arithmetical proportion.* Thus, ÷ 3 . 5 . 7 . 9 . 11 . 13, &c. is an arithmetical progression. The number by which each term exceeds the preceding one is called *the ratio* or *common difference.* The sign preceding the progression is intended to show the repetition of the intermediate terms, thus: 3 *is to* 5 *as* 5 *is to* 7, *as* 7 *is to* 9, &c.

An arithmetical progression is often called simply *an arithmetical series,* and is said to be *increasing* when the terms go on increasing, and *decreasing* when the terms decrease. Thus, ÷ 1 . 5 . 9 . 13 . 17, &c. is *an increasing,* and ÷ 17 . 13 . 9 . 5 . 1 . — 3 . — 7, &c. *a decreasing arithmetical series.* It is evident that a decreasing series is simply an increasing one reversed, and therefore that the properties of an increasing series will apply to a decreasing one, by merely substituting *minus* for *plus,* and *subtract* for *add.* We shall, therefore, consider it as increasing.

457. As the second term of the progression is formed by adding the ratio to the first, and each successive term by adding the ratio to the preceding, it is plain that any term of an arithmetical progression is composed *of the first term plus as many times the ratio as there are terms before it.* When the first term is a cipher, any term of the series is evidently composed *of as many times the ratio as there are terms before it.* Wherefore, calling the first term a, and the ratio or difference d, any term n of the series is found by adding the first term a to the product of the ratio d multiplied by the number of terms preceding n, which is $(n - 1)$, thus: $n = d(n - 1) + a$. Hence, the *ninth* term of the series ÷ 3 . 5 . 7 . 9, &c., will be $2 \times 8 + 3 = 19$. The eleventh term of the series ÷ 1 . 5 . 9, &c., is $4 \times 10 + 1 = 41$.

458. Calling the last term of the progression l, and the number of terms n, we have, according to the above, $l = d(n - 1) + a$, and consequently $d = \frac{l - a}{n - 1}$; that is to say, having the first and last terms of an arithmetical progression, and the number of terms, the *difference* or *ratio* is found in dividing the difference between the first and last terms by the number of terms diminished by a unit.

459. If we compare an arithmetical series, term for term, with the same series reversed, thus—

$$\div\ 2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \,.\, 17$$
$$\div\ 17 \,.\, 14 \,.\, 11 \,.\, 8 \,.\, 5 \,.\, 2$$

we find that as the first term goes on increasing and the last decreasing in the same ratio, the sum of the first and last terms, which stand under each other, is also the sum of any two corresponding terms of the series. The sum, therefore, of all the terms of both series, is equal to the sum of the first and last terms multiplied by the number of terms in one series; and as this product is evidently just twice the sum of the terms in one series, we infer that *the sum of all the terms of any arithmetical series is equal to half the product found in multiplying the sum of its extreme terms by the number of terms.* Hence, if we put $S =$ the sum of the series, a the first term, l the last term, and n the number of terms, we have $S = \frac{(a + l)n}{2}$.

460. In an arithmetical series the terms intervening between the first and last are said to be so many *arithmetical mean proportionals*, or simply *arithmetical means* between those two terms.

As n is the number of terms, the number of means is $n - 2$, and hence $n - 1$ is equal to the number of means plus 1; but we have seen that $d = \frac{l - a}{n - 1}$; consequently, the *ratio* is equal to the difference between the first and last terms divided by the number of means increased by a unit. Now it is evident that we may construct between any two numbers as many arithmetical means as we please, because we can always *divide their difference by the required number of means increased by a unit,* which will give the *ratio* or *common difference* of the terms; and beginning with the first term, the means are formed in regular succession *by the continual addition of the ratio.* The ratio, added to the last of the means, must of course give the last term of the progression, and this will serve as proof of the correctness of the operation. For example, if we would insert 11 arithmetical means between 1 and 61, by the formula $d = \frac{l - a}{n - 1}$, we have the ratio $d = \frac{61 - 1}{12} = \frac{60}{12} = 5$, and by continually adding 5, we have $\div 1 \,.\, 6 \,.\, 11 \,.\, 16 \,.\, 21 \,.\, 26 \,.\, 31 \,.\, 36 \,.\, 41 \,.\, 46 \,.\, 51 \,.\, 56 \,.\, 61$ for the series required. If we would insert 6 arithmetical means between 0 and 1, we have $d = \frac{1-0}{7} = \frac{1}{7}$. Hence the series is $\div 0 \,.\, \frac{1}{7} \,.\, \frac{2}{7} \,.\, \frac{3}{7} \,.\, \frac{4}{7} \,.\, \frac{5}{7} \,.\, \frac{6}{7} \,.\, 1$. To have 9 arithmetical means between 1 and 8, we have $d = \frac{8-1}{10} = \frac{7}{10}$, or 0,7; and hence the series is $\div$ 1 . 1,7 . 2,4 . 3,1 . 3,8 . 4,5 . 5,2 . 5,9 . 6,6 . 7,3 . 8.

461. Of the following five things, namely the first term a, the last l, the ratio d, the number of terms n, and the sum of the terms S, any three being given, the other two may be found by means of the two formulæ $l = d(n-1) + a$, and $S = \frac{(l+a)n}{2}$. Now, if one be unknown, we have 4 remaining; and these four (445) are susceptible of 4 combinations, 3 and 3, and hence there are four different ways of finding the value of each term, according as we take the others 3 and 3 for the given terms. First, let a be required; then the remaining terms d, l, n, s, taken 3 by 3, are susceptible of the 4 combinations, dln, lns, dns, dls. From the formula $l = d(n-1) + a$, which contains the terms of the first combination dln, we have $a = l - d(n-1)$, or $a = l - dn + d$. From the formula $s = \frac{(l+a)n}{2}$, which contains the terms of the second combination, we have $2s = nl + an$, $an = 2s - nl$, and $a = \frac{2s}{n} - l$. Again, it is plain that $l - dn + d = \frac{2s}{n} - l$; then, as the next combination is dns, we add the two equations $a = l - dn + d$, and $a = \frac{2s}{n} - l$ together, thus:

$$\begin{aligned} a &= l - dn + d \\ a &= -l + \frac{2s}{n} \\ \hline 2a &= \frac{2s}{n} - dn + d. \end{aligned}$$

by which means we eliminate l and have $a = \frac{s}{n} - \frac{dn-d}{2}$; that is, we have the value of a in the terms of the combination of given terms dns.

Lastly, to have the value of a in terms of the combination dls, as $s = \frac{(l+a)n}{2}$, we have $2s = (l+a)n$, and $n = \frac{2s}{l+a}$.

Again, $a = l - dn + d$, $dn = l - a + d$, and $n = \frac{l-a}{d} + 1$. We have, therefore, $\frac{2s}{l+a} = \frac{l-a+d}{d}$, and multiplying by $d(l+a)$, we have $2ds = l^2 - a^2 + d(l+a)$, and transposing, we have $a^2 = -2ds + l^2 + dl + da$, and $a^2 - da = -2ds + l^2 + dl$; adding, to complete the square, the square of $\frac{d}{2}$, which is half the coefficient of a, (435,) we have $a^2 - da + \frac{d^2}{4} = -2ds + \frac{d^2}{4} + dl + l^2$; extracting the square root, we have $a - \frac{d}{2} = +\sqrt{-2ds + \frac{d^2}{4} + dl + l^2}$; and transposing $-\frac{d}{2}$, we have $a = \frac{d}{2} + \sqrt{-2ds + \frac{d^2}{4} + dl + l^2}$.

In a manner similar to the above the scholar may exercise himself in verifying the respective values of the remaining terms l, d, n, s, which he will find in the following table. This table, as well as that exhibited in Geometrical Progression, is copied from the French Arithmetic of M. Bezout, which was translated into English by the author of this work, and published in the year 1825 by Samuel Wood & Sons, of New York. The tables, however, of Arithmetical and Geometrical Progression, as well as several examples which will be found in the sequel, are due to M. Peyrard, by whom many elegant additions were made to the text of M. Bezout in the abovementioned treatise.

Being given	To find	FORMULÆ for Arithmetical Progression.
$l, d, n,$	a	$a = l - dn + d.$
$l, n, s,$		$a = \frac{2s}{n} - l.$
$l, d, s,$		$a = \frac{1}{2}d + \sqrt{(-2ds + \frac{1}{4}d^2 + ld + l^2)}.$
$d, n, s,$		$a = \frac{s}{n} - \frac{dn - d}{2}.$
$a, d, n,$	l	$l = a + dn - d.$
$a, n, s,$		$l = \frac{2s}{n} - a.$
$a, d, s,$		$l = -\frac{1}{2}d + \sqrt{(2ds + \frac{1}{4}d^2 - ad + a^2)}.$
$d, n, s,$		$l = \frac{s}{n} + \frac{dn - d}{2}.$
$a, l, n,$	d	$d = \frac{l - a}{n - 1}.$
$a, n, s,$		$d = \frac{2s - 2an}{n^2 - n}.$
$a, l, s,$		$d = \frac{l^2 - a^2}{2s - a - l}.$
$l, n, s,$		$d = \frac{2ln - 2s}{n^2 - n}.$
$a, l, d,$	n	$n = 1 + \frac{l - a}{d}.$
$a, l, s,$		$n = \frac{2s}{a + l}.$
$a, d, s,$		$n = \frac{1}{2} - \frac{a}{d} + \sqrt{\left(\frac{2s}{d} + \frac{1}{4} - \frac{a}{d} + \frac{a^2}{d^2}\right)}.$
$l, d, s,$		$n = \frac{1}{2} + \frac{l}{d} + \sqrt{\left(-\frac{2s}{d} + \frac{1}{4} + \frac{l}{d} + \frac{l^2}{d^2}\right)}.$
$a, l, n,$	s	$s = \frac{an + ln}{2}.$
$a, d, n,$		$s = an + \frac{dn^2 - dn}{2}.$
$a, d, l,$		$s = \frac{a + l}{2} + \frac{l^2 - a^2}{2d}.$
$l, d, n,$		$s = ln - \frac{dn^2 - dn}{2}.$

Examples.

1. A sells to B a piece of land for $3000, which sum is to be paid in 6 years, in yearly payments, each surpassing the preceding one by a certain sum; the payment at the end of the

first year being \$400. What is the difference of the payments, and what the last payment?

$a = 400$, $n = 6$, $s = 3000$. Now by the formulæ, $d = \frac{2s - 2an}{n^2 - n} = \frac{6000 - 4800}{36 - 6} = \frac{1200}{30} = 40$, and $l = a + dn - d = 400 + 240 - 40 = 600$. Wherefore, each payment exceeds the preceding by \$40, and the last is \$600.

Proof. $400 + 440 + 480 + 520 + 560 + 600 = 3000$.

2. B buys a piece of land for \$3000, which he is to pay by yearly instalments, each exceeding the preceding one by \$40, and the first payment is to be \$400. In how many years will he clear his purchase, and what is the last payment?

$a = 400$, $d = 40$, $s = 3000$. By the formula, $l = -\frac{1}{2}d + \sqrt{(2ds + \frac{1}{4}d^2 - ad + a^2)} = -20 + \sqrt{(240000 + 400 - 16000 + 160000)} = 620 - 20 = 600$, and $n = \frac{2s}{a + l} = \frac{6000}{400 + 600} = \frac{6000}{1000} = 6$. Wherefore, he will clear his purchase in 6 years, and the last payment is \$600. In the same way may the formula be applied in the solution of all such problems.

3. B pays, for a piece of land, \$3000 by instalments, each of which exceeds the preceding one by \$40, the last payment being \$600. Required the number of payments, and the amount of the first?

Ans. Number of payments, 6; and first amount, \$400.

4. B pays, for his land, a sum in 6 payments, each of which exceeds the preceding one by \$40, the first payment being \$400. Required the cost of the land, and the last payment?

Ans. The cost, \$3000; the last payment, \$600.

5. What is the sum of the natural numbers 1, 2, 3, &c., to 1000000 inclusive? Ans. 500000500000.

6. Required to insert 9 arithmetical means between 1 and 16.

Ans. $\div 1 . 2\frac{1}{2} . 4 . 5\frac{1}{2} . 7 . 8\frac{1}{2} . 10 . 11\frac{1}{2} . 13 . 14\frac{1}{2} . 16$.

7. Required the sum of 1 . 3 . 5. 7, &c. to a million of terms.

Ans. 1000000000000.

8. What is the sum of 150 terms of a decreasing series, the first term being 397, and the common difference 4?

Ans. 14850.

9. A basket being placed on level ground, and in a right line with it, on one side, 100 stones, 1 yard apart from each other, and the first 1 yd. from the basket, how far will the

person travel who shall begin at the basket and gather the stones into it, making a separate tour for each stone?

Ans. $5\frac{65}{88}$, or very nearly $5\frac{3}{4}$ miles.

10. At what times during the twelve hours will the hour and minute hands of a common clock be together?

Ans. At $5\frac{5}{11}$ min. past one; $10\frac{10}{11}$ min. past two; $16\frac{4}{11}$ min. past 3; $21\frac{9}{11}$ min. past 4; $27\frac{3}{11}$ min. past 5; $32\frac{8}{11}$ min. past 6; $38\frac{2}{11}$ min. past 7; $43\frac{7}{11}$ min. past 8; $49\frac{1}{11}$ min. past 9; $54\frac{6}{11}$ min. past 10; and at 12.

Geometrical Progression.

462. This name is given *to a series of numbers in continual geometrical proportion.* Thus, $\div\!\!\div$ 2 : 6 : 18 : 54 : 162 : 486 is a geometrical progression. The sign $\div\!\!\div$ *is to and so is,* is used to signify the repetition of the intermediate terms, thus: 2 *is to* 6 *as* 6 *is to* 18, &c.

A geometrical progression is often called *simply a geometrical series,* and is said to be *increasing* when the numbers increase, and *decreasing* when they decrease. But as *a decreasing series is merely an increasing one reversed,* and has consequently the same properties, the same reasoning will apply to it, if we substitute *divide* for multiply, and *contains* for is contained. We shall therefore consider the series as increasing.

The number found *in dividing the greater of any two proximate terms by the less* is called the *ratio* of the progression, and as this ratio is a quotient, we shall call it q. The other elements of the progression—namely, the first and last terms, the number of terms, and sum of the series—we shall designate by the letters a, l, n, s, as in arithmetical progression.

As the second term is formed by multiplying the first term by the ratio, and each successive term, by multiplying the preceding one by the ratio, it is evident that *any term of the progression is composed of the first term multiplied by the ratio raised to a power the index of which is the number of terms less one.* Hence, the last term $l = a \times q^{n-1}$. When $a = 1$, we have $l = q^{n-1}$. If we would find the eleventh term of the above progression, we have $l = 2 \times 3^{10} = 118098$, the term required.

463. The terms intervening between the first and last are said to be *so many geometrical means between those two terms;* and any three consecutive terms being taken, the middle one is *a geometrical mean between the other two;* also, if we

peat the progression as indicated by the sign ∺, it is evident (403) that the product of the first and last of any three such terms is equal to the square of the middle one. Wherefore, to find a geometrical mean between any two numbers, *we take the square root of their product.*

Now, as $l = aq^{n-1}$; $q^{n-1} = \frac{l}{a}$, and $q = \sqrt[n-1]{\frac{l}{a}}$, that is to say, the ratio of the progression is found *in dividing the last term by the first, and extracting that root of the quotient the index of which is the number of terms diminished by a unit, or the number of means increased by a unit;* therefore, when there is one geometrical mean between two numbers, the ratio of the progression is the square root of their quotient; when there are two means, it is the cube root; when there are three, it is the fourth root of their quotient, and so on for any number of means. To produce the means, multiply the first term by the ratio, and the product again by the ratio, and so on.

464. We have seen in Evolution (366) that though we cannot always have the exact root, we can approach to it as near as we please. Hence it is evident that between two given numbers, *though we cannot always insert in integral numbers as many geometrical means as we please, we can always obtain a series of numbers which shall approach these means within any prescribed degree of exactness.*

If we would have 9 geometrical means between 3 and 3072, it is plain from the formula that the ratio $q = \sqrt[10]{\frac{3072}{3}} = \sqrt[10]{1024}$, which, by mere inspection of the table of powers, we find to be 2. Wherefore, in continually multiplying by the ratio 2, we have for the required series ∺ 3 : 6 : 12 : 24 : 48 : 96 : 192 : 384 : 768 : 1536 : 3072.

Again, if we would insert 5 geometrical means between 2 and 3, we have $q = \sqrt[6]{\frac{3}{2}} = 1{,}0699 +$, or $1{,}07 -$; and multiplying by this ratio, we have ∺ 2 : 2,14 : 2,29 : 2,45 : 2,62 : 2,80 : 3, for the required series; the mean terms being true to within a hundredth of a unit.

465. If we take any geometrical proportion $a : b :: c : d$; alternando $a : c :: b : d$; componendo $a + c : c :: b + d : d$; again, alternando $a + c : b + d :: c : d$; that is, *the sum of the antecedents of two equal ratios is to the sum of the consequents as one antecedent is to its consequent.* Again, if $c : d :: e : f$, it is plain that $a + c : b + d :: e : f$; but we

have shown that the sum of the antecedents is to that of the consequents as one antecedent to its consequent; and, therefore, $a + c + e : b + d + f :: e : f$. Hence we infer *that the sum of the antecedents of any number of equal ratios is to that of their consequents as any one antecedent is to its consequent.*

Now in the progression $\div\!\!\div 2 : 6 :: 6 : 18 :: 18 : 54$, &c., which is evidently a number of equal ratios, we easily perceive that the sum of all the antecedents comprehends *all the terms except the last,* and may be represented by $s - l$; and that the sum of all the consequents comprehends *all the terms except the first,* and may be represented by $s - a$. Now the first term being a and the ratio q, the second term is aq, and therefore $s - l : s - a :: a : aq$; and hence (403) $(s - l)aq = (s - a)a$; dividing by a, $(s - l)q = s - a$, or $qs - ql = s - a$, or $qs - s = ql - a$, or $s(q - 1) = ql - a$, or finally $s = \frac{ql - a}{q - 1}$. That is, to obtain the sum of the series, *we multiply the last term by the ratio, subtract the first term from the product, and divide the difference by the ratio diminished by a unit.*

From the two equations

$$l = a \times q^{n-1}$$

$$s = \frac{ql - a}{q - 1},$$

and in a manner exactly analogous to that pursued in constructing the table of arithmetical progression, we construct the following table, by means of which, if we know any three of the five things l, a, q, n, s, the other two can always be found with great facility. Also, with equal facility are performed, by the same table, the most difficult calculations of banking and finance, such as Compound Interest, Annuities, &c.

In the fourth section of the formula, the capital L placed before a quantity signifies the logarithm of that quantity. This section, therefore, the student may omit, till he has read the next article, in which we shall treat of those most interesting and important numbers.

Being given.	To find.	FORMULÆ For Geometrical Progression.
l, q, n	a	$a = \frac{l}{q^{n-1}}.$
l, s, n		$(s-a)\frac{1}{a^{n-1}} = (s-l)\frac{1}{l^{n-1}}.$
l, q, s		$a = lq - sq + s.$
q, n, s		$a = s\left(\frac{q-1}{q^n-1}\right).$
a, q, n	l	$l = aq^{n-1}.$
a, s, n		$(s-l)\frac{1}{l^{n-1}} = (s-a)\frac{1}{a^{n-1}}.$
a, q, s		$l = s - \frac{s+a}{q}.$
q, n, s		$l = sq^{n-1}\left(\frac{q-1}{q^n-1}\right).$
a, l, n	q	$q = \sqrt[n-1]{\frac{l}{a}}.$
a, n, s		$q^n - \frac{s}{a}q + \frac{s}{a} - 1 = 0.$
a, l, s		$q = \frac{s-a}{s-l}.$
n, l, s		$q^n - \frac{s}{s-l}q^{n-1} + \frac{l}{s-l} = 0.$
a, l, q	n	$n = 1 + \frac{Ll - La}{Lq}.$
a, l, s		$n = 1 + \frac{Ll - La}{L(s-a) - L(s-l)}.$
a, q, s		$n = \frac{L(sq - s + a) - La}{Lq}.$
l, q, s		$n = 1 + \frac{Ll - L(lq - sq + s)}{Lq}.$
a, l, n	s	$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$
a, q, n		$s = a\left(\frac{q^n-1}{q-1}\right).$
a, l, q		$s = \frac{lq-a}{q-1}.$
l, n, q		$s = \frac{l}{q^{n-1}}\left(\frac{q^n-1}{q-1}\right).$

Examples.

1. Given the last term of a geometrical series 4096, the ratio 2, and the number of terms 11, to find the first term and the sum of the series.

$l = 4096$, $q = 2$, $n = 11$. By the table

$a = \frac{l}{q^{n-1}} = \frac{l}{q^{10}} = \frac{4096}{1024} = 4$, and

$s = a\left(\frac{q^n - 1}{q - 1}\right) = 4\left(\frac{2048 - 1}{1}\right) = 4 \times 2047 = 8188.$

The first term therefore is 4, and the sum of the series 8188.

2. The first term being 4, and the ratio 2, required the 16th term: $l = aq^{n-1} = 4 \times 2^{15} = 4 \times 32768 = 131072$, the required term.

The student will easily perceive that when the series is decreasing, l and a in the formula *change places*, that is, $a = lq^{n-1}$; and hence, $l = \frac{a}{q^{n-1}}$; that is to say, instead of multiplying, *we divide the first term by the ratio raised to the power* $n - 1$.

3. Required the 9th term of a decreasing series, the first term being 295245, and the ratio 3.

$$l = \frac{a}{q^{n-1}} = \frac{295245}{6561} = 45, \text{ the term required.}$$

In a decreasing series, the second term being $\frac{a}{q}$, we have $s - l : s - a :: a : \frac{a}{q}$, whence $\frac{as - al}{q} = as - a^2$; dividing by a, $\frac{s - l}{q} = s - a$, $s - l = qs - qa$, $qs - s = qa - l$, $(q - 1)s = qa - l$, and $s = \frac{qa - l}{q - 1}$. When the series is infinite, and consequently l continually approaches 0, it is plain that $s = \frac{qa}{q - 1}$. That is, to find the sum of the series, multiply the first term by the ratio, and divide by the difference between the ratio and a unit.

4. Required the sum of the terms of the repeating decimal 0,333, &c., which is evidently an infinite series, the first term of which is $\frac{3}{10}$, and the ratio 10.

$$s = \frac{qa}{q - 1} = \frac{10(\frac{3}{10})}{9} = \frac{3}{9} = \frac{1}{3}, \text{ the known value.}$$

5. Required the sum of 8 terms of 9 terms, and of 14

terms of the decreasing series, the first term of which is 295245, and the ratio 3.

Ans. 8 terms, 442800; 9 terms, 442845; 14 terms, $442867\frac{11}{27}$.

6. Required the sum of the terms of the repeating decimals 0,999, &c., 0,4545, &c., and 0,49504950, &c.

Ans. ,999, &c. = 1. ,4545, &c. = $\frac{5}{11}$, and ,49504950, &c. = $\frac{50}{101}$.

7. What is the sum of 20 terms of the increasing series a, $3a$, $9a$, &c.? Ans. $1743392200a$.

8. What is the sum of 8 terms of the series, the first term of which is $\frac{1}{7}$, and the last $312\frac{3}{7}$? Ans. $468\frac{4}{7}$.

9. Required the sum of $\frac{7}{8}$, $\frac{7}{24}$, $\frac{7}{72}$, &c., ad infinitum.

Ans. $1\frac{5}{16}$.

10. Required to insert 10 approximate geometrical means between 7 and 2, each true within a ten-thousandth of a unit.

Ans. $\div\!\!\div$ 7 : 6,2461 : 5,5734 : 4,9731 : 4,4375 : 3,9596 : 3,5332 : 3,1537 : 2,8140 : 2,5109 : 2,2405 : 2. The scholar may omit this question till he has read the article on Logarithms.

11. Allowing 7000 gr. of wheat to the lb. avoirdupois, 60 lbs. to the bushel, and the bushel to be worth $1, what would 50 acres of land cost at 1 gr. of wheat for the first acre, 3 for the second, 9 for the third, and so on throughout?

Ans. $\$8546404615369197 48{,}536\frac{20}{2160}$.

Logarithms.

466. These artificial numbers, the invention of which is due to Baron Napier, of Scotland, are of immense utility in facilitating not only many of the most laborious numerical calculations in Navigation, Surveying, and other important branches of mathematical science, but also of those in common arithmetic, such as the calculation of the increase of population, compound interest, annuities, &c. The student will therefore see that the knowledge of their construction and application is very important. The following elucidation of this subject is a free translation from the Algebra of that lucid and elegant French writer, S. F. Lacroix.

To understand the construction of logarithms it will be necessary first to consider the nature of the exponential equation $l = aq^{n-1}$, in which we suppose n unknown. First, put $n - 1 = x$, then $l = aq^x$; whence, $q^x = \frac{l}{a}$. Put $\frac{l}{a} = y$, then $q^x = y$. Now if q be a unit, y will be a unit, whatever may be the index x. Also, if $x = 0$, we shall have $y = 1$, whatever may be the value of q. But if q be a constant quantity considerably greater than a unit, the value of y will depend entirely upon x; that is, it will increase when x increases, and decrease when x decreases; whence we may infer that x may be so varied that y shall approximate, within a prescribed degree of exactness, to any number whatever. Let x be increased, and let x' represent the new value of x, and y' the consequent new value of y. Then $q^{x'} = y'$, and as $q^x = y$, by multiplication we have $yy' = q^x \times q^{x'} = q^{x+x'}$. Hence it follows that if we know the exponents which relate to y and y', we shall, by simply adding them together, have the exponent of the product yy'.

If we divide the second equation by the first, we have $\frac{q^{x'}}{q^x} = \frac{y'}{y}$, or $q^{x'-x} = \frac{y'}{y}$.

If, of the equation $q^x = y$ we take the m^{th} power, we have $y^m = (q^x)^m = q^{mx}$. That is, the exponent corresponding to the m^{th} power of y is found in multiplying that corresponding to y by m.

Again, if of $q^x = y$, we take the n^{th} root, we have $y^{\frac{1}{n}} = (q^x)^{\frac{1}{n}} = q^{\frac{x}{n}}$, which shows that the exponent corresponding to the n^{th} root of y is found by simply dividing that corresponding to y by n.

Let us suppose, according to the above, that 10 having been taken as the constant value of q, x has been so varied that y has successively assumed the approximate value of all the natural numbers, whole and fractional, between 1 and 10000; and that a table has been constructed in which the respective values of x and y are arranged opposite each other; this is the table of ordinary logarithms which the student will find at the end of this book, and in which the values of y are the natural numbers, and those of x the corresponding logarithms.

Now from the equation $yy' = q^{x+x'}$ we see that the sum of the logarithms of any two numbers is the logarithm of their product, and hence *the multiplication of any two numbers may*, by an inspection of the table, *be performed by addition*. From the equation $q^{x'-x} = \frac{y'}{y}$ we see that the difference between the logarithms of any two numbers is the logarithm of the quotient found in dividing one by the other; and hence *division may be performed by subtraction*. From the equation $y^m = q^{mx}$ we find that the logarithm of any power of a number is found in multiplying the logarithm of the number by the index of the power; and hence *we can, by means of the table, raise a number to any power by simple multiplication.* Lastly, from the equation $y^{\frac{1}{n}} = q^{\frac{x}{n}}$ we see that we can extract any root of a number *by simply dividing its logarithm by the index of the root.*

467. The number 10, which is assumed as the constant quantity q in the calculation of ordinary logarithms, is called *the base of the system.* The actual calculation of the logarithm of any number in the scale, suppose 2, is merely the determination of x in the equation $10^x = 2$. Now as $(10)^0 = 1$, and $(10)^1 = 10$, it is evident that the logarithm of 2 is between 0 and 1, and is therefore a fraction. If we assume this fraction $\frac{1}{z}$, then $(10)^{\frac{1}{z}} = 2$, or $10 = 2^z$. Now as $2^3 = 8$, and $2^4 = 16$, it is evident that z is between 3 and 4. Suppose $z = 3\frac{1}{z'}$, and we have $10 = 2^{3+\frac{1}{z'}} = 2^3 \times 2^{\frac{1}{z'}} = 8 \times 2^{\frac{1}{z'}}$, and hence $2^{\frac{1}{z'}} = \frac{10}{8} = \frac{5}{4}$, or $2 = (\frac{5}{4})^{z'}$. Now, because $(\frac{5}{4})^3 = \frac{125}{64}$ is too small, it is plain that z' is between 3 and 4; wherefore, let $z' = 3 + \frac{1}{z''}$. Then $2 = (\frac{5}{4})^{3+\frac{1}{z''}}$, or $2 = (\frac{5}{4})^3 \times (\frac{5}{4})^{\frac{1}{z''}}$; dividing by $(\frac{5}{4})^3$, we have $(\frac{5}{4})^{\frac{1}{z''}} = 2(\frac{4}{5})^3 = \frac{128}{125} = 1{,}024$, and consequently, $\frac{5}{4} = (1{,}024)^{z''}$. Now in raising the number 1,024 successively to the 2d, 3d, and 4th, &c. power, we find that the 9th power is nearly 1,238, and as this differs from $\frac{5}{4}$ by only 0,002, it is plain that z'' is very nearly 9, though somewhat greater. Assuming $z'' = 9$, we have from the equations $z' = 3^{\frac{1}{z''}}$, and $z = 3^{\frac{1}{z'}}$, $z' = \frac{28}{9}$, and $z = 3\frac{9}{28} = \frac{93}{28}$, and hence $(10)^x = 2$ becomes $(10)^{\frac{28}{93}} = 2$. The fraction $\frac{28}{93}$ reduced, gives 0,30107 for the logarithm of 2, which is true except the last decimal figure. It is easy to see that by continuing the process, we could find the decimal value of x still nearer, continuing it to 7 decimals, we should have

$$x = 0{,}3010300.$$

To understand this value of x as that of an exponent, we must conceive that if we raise the number 10 to the 3010300th power, and extract the 10000000th root of the result, we shall have a number very near to 2; that is to say, $(10)^{\frac{3010300}{10000000}} = 2$ very nearly; the first member is somewhat greater than 2, but the number $(10)^{\frac{3010299}{10000000}}$ would be less.

468. The above method for finding the value of x direct would not be practicable, says Mons. Lacroix, when the numbers are somewhat large; but the following, given by Long, an English geometer, may be found very useful:

The determination of x in the equation $10^x = y$ being very laborious, we may proceed in an inverse order by assuming x in order to find y, and form a table of the values of y corresponding to those of x, which will afterwards enable us to determine x by means of y.

Thus, if we take x equal to all values from 0,1 to 0,9 inclusive, and first find the value of y corresponding to $x = 0,1$, which is $(10)^{\frac{1}{10}}$, the other values of y, namely, $(10)^{\frac{2}{10}}$, $(10)^{\frac{3}{10}}$, &c. will easily be found, seeing that they are merely the square, cube, 4th, 5th, &c. powers of $(10)^{\frac{1}{10}}$.

First extracting the square root, we have $(10)^{\frac{1}{2}}$, or $(10)^{\frac{5}{10}} = 3,162277660$; then, extracting the fifth root of this result, we have $(10)^{\frac{1}{10}} = 1,258925412$. By a similar process, we obtain from $(10)^{\frac{1}{10}} = 1,258925412$ the value of $\sqrt{(10)^{\frac{1}{10}}} = (10)^{\frac{1}{20}} = (10)^{\frac{5}{100}} = 1,122018454$; then taking the fifth root, we have $(10)^{\frac{1}{100}} = 1,023292992$; and by taking the square, cube, 4th, 5th, ... 9th powers of this, we obtain the values of y, corresponding to those of x, from 0,01 to 0,09.

We easily perceive that in this manner we can obtain the values of y for those of x, from 0,001 to 0,009; from 0,0001 to 0,0009, and thus form the following table:

Log.	Nat. numbers.	Log.	Nat. numbers.
0,9	7,943282347	0,00009	1,000207254
8	6,309573445	8	1,000184224
7	5,011872336	7	1,000161194
6	3,981071706	6	1,000138165
5	3,162277660	5	1,000115136
4	2,511886432	4	1,000092108
3	1,995262315	3	1,000069080
2	1,584893193	2	1,000046053
1	1,258925412	1	1,000023026
0,09	1,230268771	0,000009	1,000020723
8	1,202264435	8	1,000018421
7	1,174897555	7	1,000016118
6	1,148153621	6	1,000013816
5	1,122018454	5	1,000011513
4	1,096478196	4	1,000009210
3	1,071519305	3	1,000006908
2	1,047128548	2	1,000004605
1	1,023292992	1	1,000002303
0,009	1,020939484	0,0000009	1,000002072
8	1,018591388	8	1,000001842
7	1,016248693	7	1,000001612
6	1,013911386	6	1,000001382
5	1,011579454	5	1,000001151
4	1,009252886	4	1,000000921
3	1,006931669	3	1,000000691
2	1,004615790	2	1,000000461
1	1,002305238	1	1,000000230
0,0009	1,002074475	0,00000009	1,000000207
8	1,001843766	8	1,000000184
7	1,001613109	7	1,000000161
6	1,001382506	6	1,000000138
5	1,001151956	5	1,000000115
4	1,000921458	4	1,000000092
3	1,000691014	3	1,000000069
2	1,000460623	2	1,000000046
1	1,000230285	1	1,000000023

469. By means of this table we find the logarithm of any number, in dividing it by 10 a sufficient number of times. To obtain, for example, that of 2549, we first divide this number by $(10)^{3}$ or 1000, which is the greatest power of 10, with an integral exponent, that it contains, and we have $2549 = (10)^{3} \times 2,549$; we then seek in the table the power of 10 immediately inferior to 2,549, and we find $(10)^{0,4} = 2,511886432$, and dividing 2,549 by the last number, we have $2,549 = (10)^{0,4} \times 1,014775177$. Seeking again, in the table the power of 10 immediately inferior to 1,014775177, we find $(10)^{0,006} = 1,013911386$; then dividing the preceding quotient by this number, we have $1,014775177 \div 1,013911386 = 1,000851742$ for the third quotient. We can continue to operate thus till we find a quotient which differs from a unit only in that order of decimals which we propose to neglect.

Considering the third quotient as equal to a unit, the proposed number is resolved into factors, which are powers of 10, for we have $2549 = (10)^{3} \times (10)^{0,4} \times (10)^{0,006} = (10)^{3,406}$, whence we see that the logarithm of 2549 is 3,406. By continuing the operation till we have made 7 divisions, we shall find that this logarithm is 3,406369.

470. The following example will show that the same table serves, with still greater facility, to find a number from its logarithm:

Let 2,547 be the given logarithm, the number sought will be $(10)^{2,547} = 10^{2} \times 10^{0,5} \times 10^{0,04} \times 10^{0,007}$; it will therefore be equal to the product of the numbers

$$(10)^{2} = 100$$
$$(10)^{0,5} = 3,162277660$$
$$(10)^{0,04} = 1,096478196$$
$$(10)^{0,007} = 1,016248693$$

taken from the above table, and consequently, we shall have $2,547 = l.\,352,378$.

To find, by means of the table, the logarithm of 2, we first find that $10^{0,3} = 1,995262315$, the number nearest to 2, then as $2 \div 1,995262315 = 1,002374467$, we find the number nearest to this, and we have $(10)^{0,001} = 1,002305238$; then $1,002374467 \div 1,002305238 = 1,000069069$, and taking the number nearest to this, we have $(10)^{0,00003} = 1,000069080$, which agrees with the number 1,000069069 to the seventh decimal. But the table also shows that for one unit of difference in any decimal order of the logarithm we have a greater difference in the corresponding number; wherefore, though the logarithm 0,00003 is somewhat too great, 0,0000300 will be found true to within a ten-millionth. Wherefore, taking 1,000069069 for unity, we have $2 = (10)^{0,3} \times (10)^{0,001} \times (10)^{0,00003} = (10)^{0,3010300}$; that is to say, the logarithm of 2 is 0,3010300.

471. As in the calculation of ordinary logarithms, 10 is the base or value of q, the equation $q^x = y$, becomes $10^x = y$. Now, (466,) as the value of y depends upon that of x, if we take x, successively, equal to 0, 1, 2, 3, 4, 5, &c., the value of y will become 1, 10, 100, &c. accordingly; thus

$10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10000$, &c.

The exponent of the power to which 10 is raised, being (467) the logarithm of the number produced by the involution, we have the two following series:

Numbers,	1	10	100	1000	10000	100000	1000000, &c.
Logarithms,	0	1	2	3	4	5	6

Hence, we see that the series of logarithms is an arithmetical series, corresponding term for term with that of the numbers, which is a geometrical series. Also, that the number of units in the logarithm is the same as the number of ciphers in the corresponding power of 10.

We may here remark that the Table at the end of the book contains only the decimal part of the log. of each number as far as 10000, except the part from 1 to 100; but the integral part which should stand on the left of the comma is, from the above series, easily determined; for it is plain that the log. of any number between 100 and 1000 must be between 2 and 3; that is, must be 2 and a fraction, and hence has 2 for its integral index; the log. of any number between 1000 and 10000, must be 3 and a fraction; that is, the logarithm has 3 for its index, and so on, through every decade. Hence also *the integral index of the logarithm of any number is one unit less than the number of figures which compose that number.*

472. From the equation $yy' = q^{x+x'}$; taking $y = 100$ and $y' = 1000$, we have $100 \times 1000 = 10^{2+3}$, or $100000 = 10^5$. Hence we see that to multiply two numbers we add their logarithms, the sum of which is the log. of their product. Then, by merely inspecting the Table, we find opposite to this sum the product required. Thus, in the above series opposite, that is, above 5, the sum of the logarithms of 100 and 1000, we find 100000, which is their product.

473. Also, from the equation $q^{x'-x} = \frac{y'}{y}$, the values being

the same, we have $10^{3-2} = \frac{1000}{100}$, or $10^1 = 10$. Hence, we see that to divide one number by another, we subtract the log. of the divisor from the log. of the dividend, and the remainder is the log. of the quotient. Then, by inspection, opposite to this remainder we find the quotient sought. Thus opposite to 1, the difference between the logarithms of 100 and 1000, we find 10, which is the quotient of 1000 ÷ 100.

474. Again, let $y = 100$ and $m = 3$; then from the equation $y^m = q^{mx}$ we have $100^3 = 10^{2 \times 3}$, or $100^3 = 10^6$. Here we see that to raise a number to any power, we multiply its logarithm by the index of that power, which gives the log. of the number sought, opposite to which we find the number itself. Thus opposite to 6 we find 1000000, which is the cube of 100.

475. Lastly, let $y = 1000000$ and $n = 2$; then from the equation $y^{\frac{1}{n}} = q^{\frac{x}{n}}$, or $\sqrt[n]{y} = q^{\frac{x}{n}}$, we have $1000000^{\frac{1}{2}} = 10^{\frac{6}{2}} = 10^3$. That is, to extract any root of a quantity, we divide its log. by the index of the root; the quotient is the log. of the root required. Thus 3 is the log. of 1000, which is the square root of 1000000.

To find by the Table the Logarithm of a Number.

476. For a number consisting of three figures, we find it in the left-hand column, and opposite, in the column marked 0, is the decimal part of its log., to which we prefix the index 2. Thus, the log. of 576 is 2,760422.

When the number consists of four figures, we find the three left-hand figures as before, and opposite to these, in the column marked with the fourth figure, is the decimal part of the log., to which we prefix the index 3. Thus, the log. of 5764 is 3,760724. The log. of 576,4 is 2,760724, and that of 57,64 is, 1,760724.

477. It frequently happens in calculation that logarithms arise which are not found in the table; of such are the logarithms of fractions, mixed numbers, surd roots, &c., as also of numbers which often far exceed its limits.

To find the Logarithm of a Mixed Number.

Reduce it to an improper fraction, and (473) subtract the log. of the denominator from that of the numerator. For ex., to have the log. of $25\frac{7}{9} = \frac{232}{9}$.

From log. 232..............2,365488
we subtract *l.* 9............0,954243
and have 1,411245 for the log. required.

To find the Logarithm of a Fraction.

478. As the numerator is divided by the denominator, we should (473) subtract the log. of the denominator from that of the numerator. But as this is impossible, the denominator being the greater, we subtract the log. of the numerator from that of the denominator, and prefix to the remainder the sign —, which shows by how much the subtraction is impossible. Wherefore, operating as above, the log. of the fraction $\frac{9}{232}$ is — 1,411245.

479. To multiply by a fraction we should (472) add its log.; but, as this is negative, its addition consists (98) in merely performing the subtraction indicated by its sign. Thus, if we have 6264 × $\frac{9}{232}$, we write

l. 6264......... 3,796852
l. $\frac{9}{232}$.........—1,411245
2,385607

Having performed the subtraction indicated by the sign, we have 2,385607. Seeking this in the table, we find that it differs from the log. of 243 by only a unit of the last decimal order. Now as the logarithms of the table are only true to about half a unit of the last decimal order, we take 243 for the true result, which it is in effect.

480. To divide by a fraction we should (473) subtract its log. But to subtract a negative number is (96) to add it as a positive number. Wherefore, if we have 63 ÷ $\frac{9}{232}$, we write

l. 63........... 1,799341
l. $\frac{9}{232}$.........—1,411245
3,210586

and (96) having performed the subtraction of the negative log. by adding it as a positive number, we have 3,210586. Seeking this in the table, we find its number 1624, which is the true quotient.

481. From the series of numbers 10, 100, 1000, 10000, &c. and their logarithms 1, 2, 3, 4, &c., it is evident that to add 1, 2, 3, &c. units to the index of a logarithm is to multiply the

corresponding number by 10, 100, 1000, &c.; and *vice versâ*, to subtract from the index 1, 2, 3, &c. units, is to divide the corresponding number by 10, 100, 1000, &c.

482. Let n and $n+1$ be any two consecutive numbers, and d the difference of their logarithms. Then (473) $d = l\left(\frac{n+1}{n}\right) = l\left(1+\frac{1}{n}\right)$; that is, d approaches nearer to l. 1 or 0, according to the increase of n.

By inspecting the column marked Diff., the student will see that near the end of the table several numbers in succession have nearly the same diff. between their logarithms; that is, their logarithms successively differ by only about half a unit of the fourth decimal order.

Now suppose the difference between two consecutive logarithms to be always the same, and put $q =$ this difference. Then, as the diff. between any two consecutive numbers of the table is a unit, if a, b, c, d, &c. be consecutive numbers, and A, B, C, D, &c. their respective logarithms, the diff. between a and b is 1 unit, between a and c, 2 units, between a and d, 3 units, &c. Also the diff. between A and B is q, between A and C is $2q$, between A and D is $3q$, &c. Now as the equimultiples of any two numbers (165) have the same ratio to each other that the numbers have, it is plain that any two of the numeral differences are to one another as the differences of the logarithms of the corresponding numbers. Thus—$1:2::q:2q$; $1:3::q:3q$; $2:3::2q:3q$, &c.

483. This proportion, though not exact, is near enough, when the numbers are large for ordinary purposes, and is of great utility in finding the logarithms of numbers which exceed the limit of the table, as well as in finding the numbers from their logarithms.

Suppose, for example, we would find the log. of 458264; we separate by a comma two figures on the right, that the part on the left may be found in the table; thus, 4582,64. We then find the log. of 4582, which is 3,661055. Opposite to this, in the column marked Diff., we find 95. We then say 1 : ,64 : : 95 : 60,80, or 61. Hence, we have 3,661055 + 61 = 3,661116 for the log. of 4582,64. But as this number is 100 times too little, we add 2 units (481) to the index of its log., and have 5,661116 for the required logarithm.

Again, if we would find the number corresponding to the log. 5,661116, we first suppose 2 units to be taken from its index 5; and seeking in the table the log. next inferior to 3,661116, we find 3,661055, and its corresponding number, 4582. Then, taking the diff. between the log. found and the given one, which is 61, we say 95 : 61 : : 1 : ,64. This we append to the number 4582, which thus becomes 4582,64. Then, as this number is 100 times too little, (481,) we remove the comma, and have 458264 for the number sought.

484. From the above (476 and 481) we easily see that the log. of a decimal number is the log. of the same number considered as integral, having its index diminished by as many units as there are decimals in the number; or having for its index as many units, less one, as there are figures on the left of the comma. Thus, as the log. of 785398 is 5,895090, to have the log. of 785,398, as there are 3 decimals, we simply diminish the index 5 by 3 units, and have 2,895090 for the required logarithm. Also, it is easy to see that in this case the number of units in the index will always be one less than the number of figures remaining on the left of the comma. Hence the log. of 7,85398 is 0,895090.

485. When there are decimals only, we find the log. as before, and subtract it from as many units as there are decimals, prefixing to the remainder the sign —. Thus, to have the log. of ,785398, as there are 6 decimals, we subtract the log. of 785398 from 6, and prefixing the sign —, we have —0,104910 for the required logarithm.

486. Suppose that the given log. is such that while its index exceeds, its decimal part is found in the table; we write the corresponding number found in the table, and subtract the index of its log. from that of the given log. We then, on the right of the number found, place as many ciphers as there are units remaining in the index of the given log. Thus, if we would have the number corresponding to the log. 7,146438, we find the number corresponding to the decimal part, with 3 for its index, to be 1401. Then, as 7 — 3 = 4, we place 4 ciphers on the right of 1401, and have 14010000 for the required number.

487. When only the first figures of the decimal are found, we may proceed thus: If we would find the number corresponding to the log. 5,263584, we find in the table the next inferior decimal ,263399, supposing its index to be 3 units; hence its corresponding number is 1834. Then with the diff. be-

tween these two logarithms, which is 185, and the tabular diff. 237, we say

$$237 : 185 : : 1 : \tfrac{185}{237};$$

consequently, the corresponding number is $1834\tfrac{185}{237}$, nearly.

Now, as the proposed log. (481) belongs to a number 100 times greater, the true number is nearly $183400\tfrac{18500}{237}$; that is, $183478\tfrac{14}{237}$, or 183478,06 —.

488. As the proportion here used only approximates to the truth when the numbers are large, if the proposed log. is below that of 1500, we add to its index as many units as we can without passing the limit of the table. Having found the corresponding number, we separate as decimals, by a comma, as many figures as we have added units to the index. These decimals will often suffice, but if they do not, we find more, as above, by proportion, and annex them.

For ex.: If we would find the number corresponding to 0,624569, as this is between the logarithms of 4 and 5, and much under that of 1500, we seek it with 3 units for index, and find that is between 4212 and 4213; that is, it is 4212 and a fraction; therefore, pointing off 3 figures for decimals, we have 4,212 + for the required number. But if we would have more decimals, we take the diff. between the log. of 4,212 and the given log., which is 81; also the tabular diff., which is 103, and say $103 : 81 : : 1 : \tfrac{81}{103} = ,786+$, or ,79 —, which we annex to 4,212, and have 4,21279 — for the required number.

The logarithms of the table being only true to about half a decimal unit of the sixth order, it is not safe to push this second approximation to more than 2 places of figures.

489. When the log. is negative, we subtract it from (or rather (98) add it to) a number of units, such that the remainder may not exceed the limit of the table; and having found the number answering to this remainder, we separate on the right, by a comma, as many figures as there are units in the number, from which we subtract the log. For ex., if we have the log. — 1,724685, we say, 4 — 1,724685 = 2,275315, which answers all but the last decimal to the log. of 1885. Now the sum of any two logarithms is the log. of their product. But 4 is the log. of 1000, and — 1,724685 the log. of a fraction. Therefore, 2,275315 is the log. of the fraction multiplied by 10000; consequently, as 1885 is 10000 times too great, we place a comma on the left, and have ,1885 for the fraction sought, within a ten-thousandth.

Use of the Table.

490. The student will, by the following examples, obtain some idea of the vast utility of logarithms, especially if he proves them by the usual methods.

1. Required the quotient of 3725 ÷ 2957, within a thousandth:

$$\begin{array}{r} l.\ 3725 \ldots\ldots\ldots 3{,}571126 \\ l.\ 2957 \ldots\ldots\ldots \underline{3{,}470851} \\ 0{,}100275 \end{array}$$

Seeking the log. of the remainder with an index of 3 units, we have 1259 +; and as this is 1000 times too great, (481,) we point off 3 figures, and have 1,259 + for the required number.

2. Required the square root of 143, within a ten-thousandth.

l. 143 = 2,155336; dividing by 2, we have 1,077668, which (488) gives 11,9582 + for the required number.

3. Required the square root of $\frac{117}{119}$.

l. 119 — *l.* 117 = 2,075547 — 2,068186 = 0,007361; dividing by 2, we have — 0,003680; adding 4, we have 3,996320, which (488) gives the root ,991561, true to a millionth.

4. Required the square root of $\frac{3}{7}$.

l. 7 — *l.* 3 = 0,845098 — 0,477121 = 0,367977. Hence, *l.* $\frac{3}{7}$ = — 0,367977. Taking one-half, we have $\sqrt{\frac{3}{7}}$ = — 0,183989 —; adding 4, we have 3,816011, which (488) gives the root ,654653, true to a millionth.

5. Required the cube root of 91, within a thousandth.

l. 91 = 1,959041; dividing by 3, we have 0,653014 — which (488) gives 4,498 — for the required root. The true, root is 4,4979 +.

6. Required the seventh root of 13, within a hundred-thousandth.

l. 13 = 1,113943; dividing by 7, we have 0,159135 —, which gives the root 1,44256.

7. Required the fifth root of $\frac{7}{11}$, within a hundred-thousandth.

l. 11 — *l.* 7 = 1,041393 — 0,845098 = 0,196295. Hence, (478) *l.* $\frac{7}{11}$ = — 0,196295; dividing by 5, we have $\sqrt[5]{\frac{7}{11}}$ = — 0,039259; adding 4 units, we have 3,960741, which (488) gives the root ,91356.

8. Required the cube root of $\frac{3}{7}$.

$l.\ 7 - l.\ 3 = 0{,}845098 - 0{,}477121 = 0{,}367977$. Hence, $l.\ \frac{3}{7} = -\ 0{,}367977$. Dividing by 3, we have $l.\ \sqrt[3]{\frac{3}{7}} = -$ 0,122659; adding 4 units, we have 3,877341, which gives the required root ,753947 —, within a millionth.

9. Required the fifth root of the fourth power of 6784. This may be written thus: $\sqrt[5]{(6784)^4}$, or perhaps better thus, $(6784)^{\frac{4}{5}}$.

It is evident the log. of this root is $\frac{4}{5}$ of the log. of 6784.

$$\begin{array}{lr} \text{Wherefore, from } l.\ 6784 \ldots\ldots\ldots\ldots & 3{,}831486 \\ \text{We subtract } \frac{1}{5} \text{ of this log} \ldots\ldots\ldots\ldots & {,}766297 \\ \hline & 3{,}065189 \end{array}$$

the remainder 3,065189 gives 1161,95, the required root.

Prove by the common method.

10. Required the fourth root of the fifth power of 1161,95.

Ans. 6784.

11. Required 6 geometrical means between the numbers $5\frac{2}{5}$ and $13\frac{3}{4}$.

It is plain (463) that the ratio of the series is $\sqrt[7]{13\frac{3}{4} \div 5\frac{2}{5}}$. Hence (473 and 475) $(l.\ 13{,}75 - l.\ 5{,}4) \div 7 = l.$ of the ratio.

$$\begin{array}{lr} l.\ 13{,}75 \ldots\ldots\ldots & 1{,}138303 \\ l.\ \ 5{,}4\ \ldots\ldots\ldots & 0{,}732394 \\ \hline & 7)\ 0{,}405909 \\ \hline & {,}057987 \text{ log. of ratio.} \end{array}$$

This log. gives the ratio 1,14285. Beginning with the first term and multiplying continually by the ratio, we obtain the series $\div\!\div$ 5,4 : 6,171 : 7,053 : 8,061 : 9,213 : 10,529 : 12,033 : 13,75. Or (472) if to the log. of 5,4 we continually add the log. of the ratio, we shall successively form the logarithms of the means, viz.: 0,790381; 0,848268; 0,906355; 0,964342, 1,022329, and 1,080316, opposite to which (488) we shall find the means as above, within a thousandth.

491. When a logarithm is to be subtracted, we may add its ar. comp., taking care (100) to suppress in the result the alt unit of the comp.

12. Suppose we have 5840 ÷ 536.

to the *l.* 5840.........3,766413 we add
ar. comp. *l.* 536.........7,270835
$\cancel{1}$1,037248

and suppressing the alt, we have 1,037248, which gives the quotient 10,8955.

13. When there are several complements in one operation, we must of course suppress an alt unit for each complement. Suppose we have $\frac{683}{194} \times \frac{547}{632}$, as each denom. is a divisor we proceed thus, (478 and 100:)

l. 683.........2,834421
l. 547.........2,737987
ar. comp. *l.* 194.........7,712198
ar. comp. *l.* 632.........7,199283
$\cancel{2}$0,483889

Here, as we have 2 complements, we suppress 2 alt units, and have the log. 0,483889, which gives the required product 3,04712 —.

492. From the above we easily infer that to have the log. of a fraction, we add to the log. of its numerator the ar. comp. of the log. of its denominator; and because the alt is 10 with regard to the preceding order, we consider the index 10 units too great. Thus to have the log. of $\frac{3}{5}$, we say

l. 3.........0,477121
ar. comp. *l.* 5.........9,301030
9,778151

The sum 9,778151, considered as a positive number, is 10 units too great; we therefore employ it as such, and reject 10 at the end of the operation. For example, if we would have $\frac{3479}{217} \times \frac{3}{5}$, we say,

l. 3479.........3,541454
ar. comp. *l.* 217.........7,663540
ar. comp. *l.* $\frac{3}{5}$.........9,778151
$\cancel{2}$0,983145

As we have here employed 2 ar. complements, we suppress 2 alt units, and have for the corresponding number 9,6193, which is true to a ten-thousandth.

493. When the fraction is decimal, as the ar. comp. of its

denom. is a whole number, we add this comp. to the index of the log. of the numerator. Thus to have the comp. of the log. of ,7854, as the log. of 7854 is 3,895091, and the log. of 10000 is 4, we add its comp. 6 to the index of the log. of 7854, and have 9,895091 for the comp. required.

494. When in an operation we have made use of several ar. comps., we know (491) that the index is too great by as many tens as there are comps. used; consequently, when the index surpasses this number of tens, we can easily make the required diminution.

But should the index be below the number of tens by which it is too great, we know that it belongs to a fraction, to obtain which we seek the number corresponding to the log., and having found it, we separate by a comma as many tens of figures as there are tens too many in the index. For example, if we have 7,952341 for the result of an operation in which we have employed one ar. comp., we seek the whole number corresponding to the log. 7,952341, and find 89606875. Then, as the index is 10 units too great, we separate ten figures, and have 0,0089606875, for the required fraction.

If we do not wish so many decimals, we diminish the index, so that it may be within the limit of the table, and having found the corresponding number, we point off as many figures as there are units in the comp. of the number of figures taken from the index. Thus, in the above log. we diminish 7 by 4 units, and having found 8960, which corresponds to 3,952341, as the comp. of 4 is 6, we separate 6 figures, and have 0,00896 for the fraction sought.

495. In multiplying the comp. of a negative log. by 2, 3, 4, 5, &c., in order to have the square, cube, fourth, fifth, &c. power of its corresponding number, it is evident that the index of the resulting log. will be as many tens too great as there are units in the multiplier. Thus, if we would have the cube of ⅗, having (492) found the comp. of its log. 9,778151, we multiply by 3, and have 29,334453 for the log. of the cube, which log. is 30 units too great; wherefore, rejecting 20, and diminishing 9 by 6 units, we find the number corresponding to 3,334453, which is 2160; and, as the comp. of 6 is 4, we point off 4 figures, and have 0,216 for the required cube.

496. Also in extracting roots, when there are comps., we add to or subtract from the index, so that the number of tens by which it is too great may equal the index of the root;

then, in dividing by the index, the resulting log. will be exactly 10 units too great. For example, if we would have the fifth root of $\frac{2}{3}$:

$$\begin{array}{rr} l.\,2 \ldots\ldots\ldots\ldots & 0{,}301030 \\ \text{ar. comp. } l.\,3 \ldots\ldots\ldots\ldots & \underline{9{,}522879} \\ & 5)\underline{49{,}823909} \\ & 9{,}964782 - \end{array}$$

Having found, as usual, 9,823909 for the comp. of the log. of $\frac{2}{3}$, which is 10 units too great, as the index of the root is 5, we add 4 tens to the index of the log., that it may be 5 tens too great; and dividing by 5, we have 9,964782 for the comp. of the log. of the root. We then diminish 9 by 6 units, and having found the corresponding number, which is 9221, as the comp. of 6 is 4, we point off 4 figures, and have ,9221 for the root, within a ten-thousandth.

The fourth section of the table of Formulæ for Geometrical Progression is developed thus:

$l = aq^{n-1}$; $\frac{l}{a} = q^{n-1}$; $q^n = q\left(\frac{l}{a}\right)$; $Lq \times n = Lq + (Ll - La)$.

3d Sec. $q = \frac{s-a}{s-l}$; hence, $n = 1 + \frac{Ll - La}{Lq}$, or $n = 1 + \frac{Ll - La}{L(s-a) - L(s-l)}$.

$s = \frac{lq - a}{q-1}$; $qs - s = lq - a$; $qs - ql = s - a$; $lq = qs - s + a$; subs. for l its value aq^{n-1}; we have $q(aq^{n-1})$, or $aq^n = qs - s + a$, $q^n = \frac{qs - s + a}{a}$; $Lq \times n = L(qs - s + a) - La$, $n = \frac{L(qs - s + a) - La}{Lq}$; $q^n = q\left(\frac{l}{a}\right)$; but

1st Sec. $a = lq - sq + s$; hence, $q^n = q\left(\frac{l}{lq - sq + s}\right)$.

$Lq \times n = Lq + Ll - L(lq - sq + s)$,

$$n = 1 + \frac{Ll - L(lq - sq + s)}{Lq}.$$

BOOK VI.

STOCK—PERCENTAGE—PROFIT AND LOSS—COMMISSION AND BROKERAGE—INTEREST—PARTIAL PAYMENTS—DISCOUNT—INSURANCE—DUTIES—TARE—TAX—GENERAL AVERAGE—EQUATION OF PAYMENTS AND AVERAGE OF SALES—PARTNERSHIP—MONEYS—REDUCTION OF CURRENCIES—EXCHANGE—COMPOUND INTEREST—INCREASE OF POPULATION—ANNUITIES.

SECTION XXIV.

STOCK—PERCENTAGE—PROFIT AND LOSS—COMMISSION AND BROKERAGE—INTEREST.

Stock.

497. The capital or fund employed in trade by a merchant or company is called *stock*. Also in Book Keeping, stock signifies the owner or owners of the books.

Extensive operations, such as canals, railways, banks, &c., are usually funded by shares contributed by a Joint Stock Company. These shares are designated by the general name of *Stocks*.

The sums realized by these operations are divided amongst the stockholders according to the shares which they hold. These sums, which are called *dividends*, are commonly declared annually or semi-annually; and, according to the amount of the dividends, the stocks or shares rise or fall in value. If they neither rise nor fall, but remain at their original cost, they are said to be *at par*. When they rise, they are *above par*, and are sold at so much per cent. *premium* or *advance*. When they fall below their first cost, they are *below par*, and are sold at so much per cent. *discount*.

Ex. 1. At 7½ p. c. advance, what is the worth of $2500 Company Stock?

To find the advance, we take 7½ p. c. of $2500, thus:

5 p. c., or $\frac{1}{20}$ of 2500 is 125 or 10 p. c. is 250
2½ " $\frac{1}{40}$ 62,50 Subtr. ¼ 62,50
187,50 187,50

Or, by any of the following: 25 × 7½ = 175 + 12,50 = 187,50; 7,5 × 25 = 750 ÷ 4 = 187,50; 15 × 12½ = 180 + 7½ = 187,50; 30 × 6¼ = 180 + 7½ = 187,50. Then 2500 + 187,50 = $2687,50. Answer.

As. 1 p. c. is $\frac{1}{100}$, 5 p. c. is $\frac{5}{100} = \frac{1}{20}$; 20 p. c. is $\frac{20}{100} = \frac{1}{5}$; 25 p. c. is $\frac{25}{100} = \frac{1}{4}$, &c.

2. What is the worth of $2800 stock, at 15¾ p. c. discount? 15¾ p. c. of 2800 = 28 × 15¾ = 63 × 7 = 441; then 2800 — 441 = $2359. Answer.

3. At 9⅝ p. c. discount, what is the value of 32 shares Bank Stock, the par value of a share being $275? Ans. $7953.

4. What is the value of $10500 Railway Stock, at 13⅔ p. c. advance? Ans. $11935.

5. At 28¾ p. c. below par, what is the worth of $6750 Stock? Ans. $4809,37½.

6. What is the value of £5600 Stock, at 5¾ p. c. premium? Ans. £5922.

7. What is the value of £6795 6 s. 8d. Stock, at 8⅔ p. c. discount? Ans. £6206 8s. 5¾⅕ d.

Percentage.

498. This is an allowance of so much per cent. (per centum, Lat. by the hundred,) used in most commercial calculations, such as Interest, Commission, Discount, Profit and Loss, the rise and fall of Stocks, prices of goods, &c.

It is also frequently employed as a term of comparison to express the superiority or inferiority of an article; also to estimate the increase or diminution of quantities, population, &c.

The hundredth part or 1 p. c. of an integral number is found in pointing off two figures; also, of a decimal number, in moving the comma two places towards the left; consequently, in multiplying this result by 2, 3, 4, 5, &c., we obtain 2, 3, 4, 5, &c. p. c. of the given number.

Or, which is the same thing, to have 1, 2, 3, 4, 5, &c. p. c. of any number, we multiply that number by ,01, ,02, ,03, 04, ,05, &c. Thus 7 p. c. of $985 is 9,85 × 7, or 985 × ,07 = $68,95.

499. Merchants often mark their goods from the Invoice or Bill of Cost and Charges, so as to realize upon the first or *prime cost* a profit of so much per cent. This varies greatly in different kinds of business.

Suppose an article has cost $2,40; what must we mark it, so as to gain 20 p. c.?

100 : 2,40 : : 20, or 5 : 2,40 : : 1 : ,48; wherefore,
2,40 + ,48 = $2,88. Answer.

Here we see that we must multiply the prime cost by the rate p. c., divide by 100, and to the prime cost, add the result.

Or thus: 100 : 120 : : 2,40, or 5 : 6 : : 2,40 : $2,88. Answer. That is, add the percentage to 100; multiply by the prime cost, and divide by 100.

If we would mark the above article so as to lose 20 p. c., we proceed in the same manner, except that instead of adding we subtract; that is, by the first method, instead of $2,40 + ,48, we have 2,40 — ,48 = $1,92. Answer. Operating by the second method, we subtract the percentage, and say

100 : 80 : : 2,40, or 5 : 4 : : 2,40, or 1 : 4 : : ,48 : $1,92. Ans.

Ex. 1. What must the following prices be respectively raised to, so as to gain 15 p. c.: namely, $8,50, $9,20, $10,80, $11,60, $12,50, $15,80, $16,50, and $18,40?

Ans. $9,77½, $10,58, $12,42, $13,34, $14,37½, $18,17, $18,97½, and $21,16.

2. How much p. c. must each of the preceding answers be reduced, to give the prime cost or number which gave it?

Ans. $13\frac{1}{23}$ p. c.

500. If A be increased 15 p. c. it will be $A+\frac{15A}{100}=\frac{115A}{100}=\frac{23A}{20}$. Now (188) the ratio of this to A is $\frac{23}{20}$, and its reciprocal $\frac{20}{23}$. Therefore, having advanced the price of an article 15 p. c., if we multiply the result by $\frac{20}{23}$, we shall (219) obtain the prime cost. Also, if we multiply 15 by $\frac{20}{23}$, we shall obtain the percentage of discount. Thus, $15\times\frac{20}{23}=\frac{300}{23}=13\frac{1}{23}$.

Hence, we easily deduce the rule: Add the advance percentage to 100, divide the result by 100, reduce the terms, and multiply the rate p. c. by the reciprocal of the result, which will give the percentage of discount, to reduce again to prime cost.

3. Advance the following prices 12 p. c.: namely, 4 s. 2 d., 6 s. 3 d., 8 s. 4 d., 10 s. 5 d., 12 s. 6 d., 14 s. 7 d., 16 s. 8 d., and 18 s. 9 d.

Ans. 4 s. 8 d., 7 s., 9 s. 4 d., 11 s. 8 d., 14 s., 16 s. 4 d. 18 s. 8 d., and 21 s.

4. How much p. c. must the preceding advanced prices be reduced to give the prime cost? Ans. $10\frac{5}{7}$ p. c.

5. What is 6 p. c. of \$20000? Ans. \$1200.

6. What is $19\frac{1}{3}$ p. c. of \$40000? Ans. \$7733,$33\frac{1}{3}$.

7. What is $15\frac{5}{6}$ p. c. of \$56789? Ans. 8991,$59\frac{1}{6}$.

8. Having received, on \$56789 Stock, an advance of $15\frac{5}{6}$ p. c., at what discount must we sell the whole amount so as neither to gain nor lose? Ans. $13\frac{93}{139}$ p. c.

9. What is $13\frac{98}{139}$ p. c. of \$65780,$59\frac{1}{6}$?

Ans. \$8991,$59\frac{1}{6}$.

501. To find how many p. c. one number A is of another number B: First, 1 p. c. of B is $\frac{B}{100}$; then the number of times that A contains this one p. c. is evidently the number sought. We, therefore, divide A by $\frac{B}{100}$, or multiply A by $\frac{100}{B} = \frac{A}{B} \times 100$. Hence it is plain that to find how many p. c. one number is of another, we multiply the ratio of that one to that other by 100.

10. What p. c. of $\frac{9}{10}$ is $\frac{7}{18}$? Ans. $48\frac{11}{18}$.

11. How much is $48\frac{11}{18}$ p. c. of $\frac{9}{10}$? (498.) Ans. $\frac{7}{18}$.

12. How much p. c. must be taken from $\frac{9}{10}$ so as to reduce it to $\frac{7}{18}$? (500.) Ans. $51\frac{7}{18}$.

13. What p. c. of £20 is 6 s. 8 d? Ans. $1\frac{2}{3}$ p. c.

14. What p. c. of \$116,80 is \$7,20? Ans. $6\frac{12}{73}$ p. c.

15. What p. c. of £50000 is £150, 13 s. 4 d.?

Ans. $\frac{113}{375}$, or not quite $\frac{1}{3}$ p. c.

502. To find what percentage of one number will equal a given percentage of another:

Let the given number be A and B, and let the question be, What p. c. of A will equal *q* p. c. of B? Now, if A be multiplied by $\frac{B}{A}$, the product will be B. Wherefore (165) any percentage or part of A, multiplied by the same ratio, will give the same part or percentage of B. Hence we have this rule: Place that number, the percentage of which is given, over the other, and multiply the given percentage by this fraction reduced to its lowest terms.

16. What per centage of £17680 will equal 16 p. c. of £96470? Ans. $87\frac{67}{221}$.

Profit and Loss.

503. This, in a merchant's books, is the title of an account of incidental or accidental expenses, gains or losses. We shall here consider the subject as the gain realized and loss sustained by trading, which are generally estimated at a certain rate p. c. on the amount of capital employed.

To find the gain p. c.: 1. When with a sum A we have realized a greater sum B, as 1 p. c. of A is $\frac{A}{100}$, we divide the gain (B — A) by this fraction, which will evidently give the gain p. c.

2. Or, because A : B : : 100 : 100 + gain p. c.

Divide the sum realized by the sum invested, reduce the fraction, multiply by 100, and suppress 100 in the result.

Ex. 1. Suppose that with \$14280 we have realized \$18460, what is the gain p. c.?

$\frac{100}{14280} = \frac{5}{714}$, reciprocal of $\frac{A}{100}$, into which we involve the gain (B — A) = 18460 — 14280 = 4180; thus, $\frac{5 \times 4180}{714} = \frac{20900}{714} = 29\frac{97}{357}$, required gain p. c. Or thus: $\frac{1846|0}{1428|0} = \frac{923}{714}$; then $\frac{923 \times 100}{714} = £29\frac{97}{357}$ p. c. as before.

2. Suppose that with a capital of \$10000 we have realized \$16580, what is the gain p. c.?

$\frac{A}{100} = 100$; B — A = 6580, dividing by 100, we have 65,8, or $65\frac{4}{5}$ gain p. c. Or $\frac{16580}{10000} \times 100 = 165{,}80$; suppressing 100, we have 65,80 = $65\frac{4}{5}$, as before.

3. Having invested £15600 in stocks, and subsequently sold the same for £25000, what is the gain p. c.?

Ans. $60\frac{10}{39}$ p. c.

4. What is $60\frac{10}{39}$ p. c. of £15600? Ans. £9400.

504. When there is a loss, that is, when B is less than A, we have the proportion A : B : : 100 : 100 — loss p. c.; consequently, in operating by rule 2, instead of suppressing 100 in the result, we must subtract the result from 100.

5. Invested £25000 in stocks, and sold for £15600; what is the loss p. c.?

Rule 2. $\frac{15600}{25000}=\frac{156}{250}=\frac{78}{125}$; $\frac{78\times 100}{125}=\frac{78\times 4}{5}=\frac{312}{5}$ $=62\frac{2}{5}$, which is 100 — loss p. c.; therefore, $100-62\frac{2}{5}$ $=37\frac{3}{5}$ p. c. is the required loss.

6. What is $37\frac{3}{5}$ p. c. of £25000? Ans. £9400.

505. Also, when the investment and whole gain or loss are simple integral numbers, we may place the gain or loss over the investment in the form of a fraction, reduce to lowest terms, multiply by 100, and reduce the result for the required percentage.

7. Invested $23500, and sold for $16500, what was the loss p. c.? Ans. $29\frac{37}{47}$.

8. Bought 20 shares Bank Stock at $350 a share; sold 12 shares at $14\frac{2}{7}$ p. c. discount, and the remainder at $28\frac{4}{7}$ p. c. discount; what was the loss p. c. on the whole?
Ans. 20 p. c.

9. Bought 26 bales of goods at $26 per bale; found 3 bales so damaged that they were sold 15 p. c. below cost; how much p. c. profit must be charged on the remainder, so as to gain 20 p. c. on the whole? Ans. $24\frac{13}{23}$ p. c.

10. What is $24\frac{13}{23}$ p. c. of $598? Ans. $146,90.

11. What p. c. of $811,20 is $744,90? Ans. $91\frac{43}{52}$.

12. What p. c. of $811,20 is $66,30? Ans. $8\frac{9}{52}$.

Commission and Brokerage.

506. Commission is a charge of so much p. c. made by agents, factors, brokers, or correspondents for transacting business on behalf of others.

The Commission charged by a Broker is called *Brokerage.*

Ex. 1. A factor buys on my account goods to the amount of $4569,50; what is his commission at $2\frac{1}{2}$ p. c.
Ans. $114,23$\frac{3}{4}$.

2. An agent sells goods to the amount of £569 11 s. 6 d.; what is his commission at $4\frac{1}{2}$ p. c.? Ans. £25 12 s. $7\frac{1}{2}$ d.—.

3. My correspondent makes sales of merchandise on my account to the amount of $12600, commission $2\frac{1}{4}$ p. c.; how much does he place to my account? Ans. $12316,50.

4. Having sold 18 cwt. 2 qr. 14 lb. of cochineal for my correspondent at $86 per cwt., commission $5\frac{3}{4}$ p. c., how much must I pay over? Ans. $1509,65.

Interest.[*]

507. This is a percentage paid for the use or retention of money. When no time is specified, one year is understood. Thus, when we say 5 p. c., the words *per annum*, by the year, are understood.

The sum on which interest is paid is called *the principal.* The interest of 100, for one year, is called *the rate*, and the sum of principal and interest, *the amount.*

Interest is usually regulated by law,[†] and varies considerably in different countries, according to the abundance or scarcity of money, facilities of trade, &c.

Ex. 1. Suppose we would have the interest of $1776 for one year, at 6 p. c.

The question here evidently is, if $100 gain $6, what will $1776 gain? Consequently, we have the proportion

$100 : $1776 : : $6 : $106,56. Answer.

Hence the general rule: Multiply the principal by the rate p. c., and divide by 100. Or, as the question (498) evidently requires nothing more than 6 p. c. of $1776, we multiply by ,06, thus: $1776 × ,06 = $106,56, as before.

That is, *multiply the principal by the rate considered as hundredths.*

508. If we require the interest for 2, 3, 4, 5, 6, &c. years, it is plain that we have only to multiply $106,56, which is the interest for one year, by 2, 3, 4, 5, 6, &c. Or, we may multiply the rate by the number of years, (419,) and proceed as before.

Suppose we would have the interest of $1776 for 3½ years, at 6 p. c., we say, ,06 × 3½ = ,21; then, $1776 × ,21 = $272,96.

Proof. $106,56 × 3½ = $53,28 × 7 = $372,96.

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[*] "In the New England States, New Jersey, Pennsylvania, Delaware, Maryland, Virginia, North Carolina, Tennessee, Kentucky, Ohio, Indiana, Illinois, Missouri, Arkansas, and the Dist. of Columbia, and on U. S. Notes, the rate is 6 p. c. In New York, S. Carolina, Michigan, Wisconsin, and Iowa, it is 7 p. c. In Georgia, Alabama, Mississippi, and Florida, 8 p. c. In Louisiana, 5 p. c. though the bank interest is ,06, and conventional interest may be as high as ,10. In Maryland the interest on tobacco contracts is ,08. In Mississippi, Missouri, and Arkansas, the interest by agreement may be as high as ,10, and in Illinois, Wisconsin, and Iowa, as high as ,12."—*Chase's Arith.*, page 66.

[†] In England, legal interest is 5 p. c.

509. Suppose we would have the int. for months, say 8 mos.; the proportion is then compound, and becomes

$\$100 : \$1776 :: \$\cancel{6} : \$71,04$

$\cancel{12} : \quad \cancel{8}$

$\cancel{2}$ 4 half the number of months, which divided by 100 is ,04.

Hence we see that the int. at 6 p. c. for any number of months is found in multiplying the principal by half the number of months, considered as hundredths.

The following table shows the number of days from the beginning of the year to the first of each respective month :

1 mo. (Jan.)	0	5 mo. (May)	120	9 mo. (Sept.)	243
2 mo. (Feb.)	31	6 mo. (June)	151	10 mo. (Oct.)	273
3 mo. (Mar.)	59	7 mo. (July)	181	11 mo. (Nov.)	304
4 mo. (Apr.)	90	8 mo. (Aug.)	212	12 mo. (Dec.)	334

By this table we easily find the number of days from any day of one month to any day of another.

Rule : Find the difference between the numbers opposite the given months, under which write the difference between the given dates, and when the first date is the smaller number, add ; when it is the greater, subtract. To the result add 1, when the second month of leap-year is included between the given dates.

When the last date is in the following year, reverse the order of the dates, proceed as above, and subtract the result from 365. The remainder is the required number. Also, to this remainder add 1, when the second month of leap-year is included between the given dates.

1. Required the number of days from 4 mo. 12 to 12 mo. 25 ?

334 — 90 = 244 No. of days from 4 mo. 1 to 12 mo. 1.
25 — 12 = 13 diff. of dates added.
257 Answer.

2. What is the number of days from Apr. 25 to Dec. 12 ? Ans. 231.

3. Required the number of days from 12 mo. 25, 1853, to 4 mo. 12, 1854 ? Ans. 108.

4. Required the number of days from 12 mo. 12, 1855, to 4 mo. 25, 1856 ? Ans. 135.

510. As ,06 is the int. of 100 for one year at 6 p. c., $\frac{1}{12}$ of

,06 = ,005 is the int. for 1 mo.; and $\frac{1}{5}$ of ,005 = ,001 is the int. for $\frac{1}{5}$ of 1 mo. or 6 days. Consequently, the int. for any number n of days $6 : n :: ,001$; multiplying (413) the first and third terms by 1000, we have $6000 : n :: 1 : \frac{n}{6000}$. Hence we have the following rule, called the bank rule:

To find the int. at 6 p. c. for any number of days, multiply the principal by the number of days, and divide by 6000.

Ex. 1. Suppose we would have the int. of \$1776 for 24 days, we say, $\frac{1776 \times 24}{6000} = 1,776 \times 4 = \$7,104$. Answer.

To find the Simple Interest of any amount for years, months and days:

Reduce the years and months to months, place $\frac{1}{3}$ the number of days on the right, and multiply the result, considered as mills, by half the principal, or half the result by the whole principal.

Examples.

1. Required the interest of \$625 for 2 yrs. 9 mo. 21 d.

2 yrs. 9 mo. = 33 mo. $21 \div 3 = 7$; then $337 \times 312\frac{1}{2} = 105312\frac{1}{2}$ m. = \$105,31$\frac{1}{4}$. Or, $,1685 \times 625 = 1685 \div 16 = \$105,31\frac{1}{4}$, (273.)

Elucidation.

21 d. = $\frac{21}{30}$ mo. = $\frac{7}{10}$ mo.; consequently, these tenths have their true value when placed on the right of 33 mo. Then

$$\left.\begin{array}{l}100 : 625 :: 6\\ \;\;12 : 33,7\end{array}\right\}$$ Now the rate 6 p. c. measures the constant divisor 12 and the result is 2.

Also the term 337 tenths, divided by the constant divisor 100, becomes 337 thousandths or mills, hence the rule.

2. What is the int. of \$1337,67 for 3 yrs. 5 mo. 17 d.?

3 yrs. 5 mo. 17 d. $\div 100 = ,415\frac{2}{3}$ mo.;—then half the principal $= 668,835 \times ,415\frac{2}{3} = \$278,01\frac{483}{2000}$. Or $222,945 \times 1,247 = \$278,01 +$.

3. What is the int. of £68 15 s. 8 d. for 4 yrs. 3 mo. 19 d.?

£68 15 s. 8 d. = 68,78$\frac{1}{3}$: then $,516\frac{1}{3} \times 34,39\frac{1}{6} = 1,549 \times 11,46\frac{7}{18}$ = £17,7575638 = £17 15 s. 1$\frac{3}{4}$ d. +.

511. As the int. at 6 p. c. is found with great facility, the int. at any other rate may be found by Practice. Thus, for the int. of \$1776 at 4$\frac{1}{2}$ p. c., we say

3 p. c. is $\frac{1}{2}$) 106,56 int. at 6 p. c.

1$\frac{1}{2}$ " is $\frac{1}{2}$) 53,28 or, by complement, \$106,56

26,64 for 1$\frac{1}{2}$ p. c. subtr. $\frac{1}{4}$ 26,64

\$79,92 \$79,92.

2. What is the int. of \$236,14 at 6 p. c.?
Ans. \$14,17 —.

3. What is the int. of \$993,60, at 4$\frac{1}{2}$ p. c.?
Ans. \$44,71.

4. What is the int. of £294 16 s. 8 d. at 5 p. c.?
Ans. £14 14 s. 10 d.

5. What is the int. of \$900 at 7 p. c.? Ans. £63.

6. What is the int. of \$1358 at 6 p. c. for 3$\frac{1}{2}$ years?
6 × 3$\frac{1}{2}$ = 21, and 13,58 × 21 = Ans. \$285,18.

7. What is the int. of \$169, for 7 mo. at 6 p. c.?
\$1,69 × 3$\frac{1}{2}$ = Ans. \$5,91$\frac{1}{2}$.

8. What is the int. of \$2600, for 3 yrs. 7 mo. and 5d. at 6 p. c.? \$26 × 21$\frac{7}{12}$ = Ans. \$561,16$\frac{2}{3}$.

9. What is the int. of £65 13 s. 4 d., for 78 days at 6 p. c.?
,065$\dot{6}$ × 13 = Ans. 17 s. 0$\frac{22}{25}$ d.

10. What does £786 16 s. amount to in 4 yrs. 8 mo. 15 d., at 6 p. c.

7,868 × 28$\frac{1}{4}$ = 1,967 × 113 = 222,271 = £222 5 s. 5 d. +. Then £786 16 s. + £222 5 s. 5 d. = Ans. £1009 1 s. 5 d. +.

512. To find in what time a sum at simple int. will double itself—that is, in how many years the whole amount of int. will equal the principal—divide 100 by the rate p. c.

1. Suppose we would find in what time the int. of \$350 will amount to \$350, at 6 p. c., we say

$$\overset{\text{Cap.}}{100} : \overset{\text{Cap.}}{350} :: \overset{\$}{6} : \overset{\$}{21}, \text{ one year's int.}$$

$$\text{then } \overset{\text{Int.}}{21} : \overset{\text{Int.}}{350} :: \overset{\text{Yr.}}{1} : \tfrac{350}{21} = 16\tfrac{2}{3} \text{ yrs. Answer.}$$

But (406) $\frac{100}{6} = \frac{350}{21}$; hence the rule, according to which a sum at 7 p. c. will double itself in 14$\frac{2}{7}$ years, at 8 p. c. in 12$\frac{1}{2}$ yrs., at 9 p. c. in 11$\frac{1}{9}$ yrs. &c.

513. To find in what time the int. of a given sum will attain any other given amount: Divide the given amount by the int. of the given sum for one year at the given rate.

2. In what time will the int. of \$350 amount to \$200 at 6 p. c.?

$$\overset{\text{Int.}}{21} : \overset{\text{Int.}}{200} :: \overset{\text{Yr.}}{1} : 9\tfrac{11}{21} \text{ yrs. Ans.}$$

Hence the rule.

SECTION XXV.

PARTIAL PAYMENTS — DISCOUNT — INSURANCE — DUTIES — TARE—TAX—GENERAL AVERAGE.

Partial Payments.

514. When these payments, which are made at sundry times to cancel a Promissory Note, or other obligation, are, with their respective dates, entered on the back of such instruments, they are called *endorsements.* When a final settlement is made within one year from the date of the instrument, it is customary to calculate the interest on the whole from the time of its date, or from the time it becomes due, to the time of settlement; and on each endorsement, from its date, in like manner. Then the amount of all the endorsements, with their several interests, is subtracted from that of the principal and its interest, and the remainder is the balance due.

Portland, Mar. 10, 1853.

For value received, I promise to pay Rufus Horton, or order, five hundred dollars, on demand, with interest.

$500. Noble H. Heath.

Endorsements.—June 5, 1853, received $60. July 20, 1853, received $95. Sept. 19, 1853, received $100. Nov. 23, 1853, received $40. What is due Mar. 10, 1854?

Amount of $500 for 1 yr. $530,00.

	Int. of $60	for 9 mo. 5 d.....	$2,75	
	95	" 7 mo. 18 d.....	3,61	
$295,00	100	" 5 mo. 19 d.....	2,82	
9,88	40	" 3 mo. 15 d.....	0,70	304,88
$304,88	$295		$9,88	$225,12

Balance due Mar. 10, 1854, $225,12.

Gorham, Jan. 4, 1853.

For value received, I promise to pay John Horton, or order, one thousand, two hundred and sixty dollars, on demand, with interest. Philip Heath.

$1260.

Endorsements.—Mar. 15, 1853, received $200. June 9, 1853, received $130. July 20, 1853, received $400. Sept. 15, 1853, received $300. What is due Jan. 4, 1854.

Ans. $274,97.

WINDHAM, Feb. 6, 1853.

For value received, I promise to pay Amos Hanson, or order, eight hundred and forty dollars, on demand, with interest.

$840. MARK KNIGHT.

Endorsements.—May 16, 1853, received $180. July 8, 1853, received $100. Aug. 19, 1853, received $60. Oct. 14, 1853, received $250. What was due Dec. 12, 1853?

Ans. $280,54.

515. Though the preceding is, by some, considered the only equitable rule, the following is the legal one, as stated in the N. Y. Chancery Reports, (see Emerson's Third Part, page 104:)

"The rule for casting interest, when partial payments have been made, is to apply the payments, in the first place, to the discharge of the interest then due. If the payment exceeds the interest, the surplus goes towards discharging the principal, and the subsequent interest is to be computed on the balance of principal remaining due. If the payment be less than the interest, the surplus of interest must not be taken to augment the principal; but interest continues on the former principal until the period when the payments, taken together, exceed the interest due, and then the surplus is to be applied towards discharging the principal, and interest is to be computed on the balance, as aforesaid."

NEW YORK, July 8, 1850.

For value received, I promise to pay Hyacinthe Peugnet, or order, one thousand five hundred dollars, on demand, with interest. JEAN MANESCA.

$1500.

Endorsements.—Nov. 19, 1851, received $200. Oct. 16, 1852, received $50. July 11, 1853, received $45. Oct. 20, 1853, received $560. What was due Dec. 12, 1853?

Principal......	$1500,00
Int. from July 8, 1850, to Nov. 19, 1851, (1 yr. 4 mo. 11 d.,)..	122,75
Amount......	1622,75
First pay't exceeding int.......	200,00
New principal......	1422,75

Int. from Nov. 19, 1851, to Oct. 16, 1852, (10 mo. 28 d.,)	$77,78	
Sec'd pay't less than int.	50,00	
Balance of int. due	27,78	
Int. from Oct. 16, 1852, to July 11, 1853, (8 mo. 25 d.,)	62,85	
	90,63	
Third pay't less than int.	45,00	
Balance of int. due	45,63	
Int. from July 11, 1853, to Oct. 20, 1853, (3 mo. 9 d.,)	23,47	$69,10
Amount due Oct. 20, 1853		1491,85
Fourth pay't exceeding int.		560,00
New principal		931,85
Int. from Oct. 20, 1853, to Dec. 12, 1853, (1 mo. 22 d.,)		8,07
Amount due Dec. 12, 1853		$939,92

WESTBROOK, Feb. 20, 1850.

For value received, I promise to pay Jeffery Smedley, or order, one thousand dollars, on demand, with interest.

$1000. NOBLE HEATH, JR.

Endorsements.—Oct. 15, 1850, received $150. Nov. 3, 1851, received $50. Aug. 26, 1852, received $40. Jan. 5, 1853, received $45. June 9, 1853, received $300. Sept. 29, 1853, received $400. How much was due on this Note, Mar. 11, 1854? Ans. $212,07.

Discount.

516. This is an allowance of so much p. c. for the payment of money before it is due, or for prompt payment in the purchase of goods on which credit is allowed. The latter is sometimes called Rebate.

As $100 at 6 p. c. amounts, in one year, to $106, it is plain that $100 is the present worth of $106, payable at the end of that time. Hence the ratio of any sum, due at the end of one year at 6 p. c., to its present worth, is $\frac{106}{100} = \frac{1,06}{1}$; wherefore (219) any sum multiplied by $\frac{1}{1,06}$, which is the reciprocal of this ratio; that is, divided by 1,06, will give its present worth for one year at 6 p. c.

Ex. 1. If A has B's Note for $583, due in one year, without int., how much should A allow B for prompt payment, discount at 6 p. c.?

$\$\frac{583}{1,06} = \$550$ present worth; then
$583 - 550 = \$33$ discount, which A allows B.

Proof. $550 \times ,06 = \$33$ int. and $\$550 + 33 = 583$ Am't of Note.

517. As $\frac{106}{100}$ is the ratio of any sum due at the end of one year to its present worth; and the rate 6 p. c., the discount of 106, which will give the present worth, it is plain that 106 : 6 : : any sum : its discount at 6 p. c.

Hence the rule: Multiply any given sum by the rate p. c. and divide by 100 plus that rate.

2. Thus for the discount of the above note, we say,

$$\frac{583 \times 6}{106} = \frac{583 \times 3}{53} = 11 \times 3 = \$33, \text{ required discount.}$$

Required the discount of each of the following sums:

3. $840 at 5 p. c. = $\frac{840}{21}$ = Ans. $40.
4. $963 at 7 p. c. = 9 × 7 = ... 63.
5. $1080 at 8 p. c. .... 80.
6. £112 at 5 p. c. .... £5 6 s. 8 d.
7. £112 at 7 p. c. .... £7 6 s. $6\frac{1}{2}$ d. + $\frac{2}{107}$ qr.
8. £967 16 s. 11 d. at 7 p. c. .... £63 6 s. $4\frac{9}{107}$ d.
9. £1056 17 s. $11\frac{3}{4}$ d. at 8 p. c. .... £78 5 s. $9\frac{1}{4}$ d. $\frac{7}{27}$ qr.

Required the present worth of each of the following sums:

10. $100 at 7 p. c. .... Ans. $93,46—.
11. $546,50 at 8 p. c. Prove as in Ex. 1....... $506,01$\frac{23}{27}$.
12. £967 16 s. 11 d. at 7 p. c. .... £904 10 s. $6\frac{98}{107}$ d.
13. £1056 17 s. $11\frac{3}{4}$ d. at 8 p. c. ..... £978 12 s. $2\frac{1}{4}$ d. $\frac{20}{27}$ qr.

518. Though the preceding is undoubtedly the equitable or true rule of Discount, it is by no means that which is generally practiced. The Bank Rule is to deduct the interest of the given sum for the given rate and time.

Thus, if, where the int. is 6 p. c., we present a Note of $500 for discount, the Bank deducts $30, which is the int. of $500 for one year at 6 p. c. and we receive $470 for the Note. This, therefore, needs no further elucidation.

By the true rule, the present worth is the sum which, at int. at the given rate, will reach the amount of the Note. But we find that the int. of $470 at 6 p. c. is only $28,20, and the amount $498,20, instead of $500.

519. The amount of discount being given, to find the true

rate. Let q = the rate; D, the discount; P, the present worth, and S, the amount of Note. Then by the true rule, (517,)

$100 + q : q :: S : D$ dividendo, (410.)
$100 : q :: P : D$ alternando, (408.)
$100 : P :: q : D$ invertendo, (408.)
$P : 100 :: D : q$; that is,

The present worth is to 100, as the given discount to the rate.

Ex. 1. Having, for a Note of $500, received $470; what is the true rate of discount?

$470 : 100 :: 30 : 6\frac{18}{47}$ p. c. the required rate.

Proof. $\$4{,}70 \times 6\frac{18}{47} = {,}1 \times 300 = \$30$.

520. The Bank rate, being merely the int., is found thus:

$5|00 : 1|00 :: 30 : 6$ p. c.

Hence, we have the following rule:

For the true rate, divide 100 times the discount by the present worth. For the Bank rate, divide 100 times the disc. (or rather int.) by the whole sum.

2. Having at a bank received $558 for a Note of $600, what is the true, as well as the Bank, rate of discount?

Ans. True rate $7\frac{49}{93}$ p. c.; Bank rate, 7 p. c.

Prove thus: $5{,}58 \times 7\frac{49}{93} = {,}06 \times 700 = 6 \times 7 = 42$. Also, $6{,}00 \times 7 = 42$.

3. Having discounted a Note of $720, for which was received $662,40, what is the rate p. c.?

True rate, $8\frac{16}{23}$ p. c.; Bank rate, 8 p. c.

4. Having for a Note of £575 5 s., received £546 9 s. 9 d., what is the rate of discount?

True rate, $5\frac{5}{19}$ p. c.; Bank rate, 5 p. c.

Insurance.

521. This is a bond or contract between two parties by which one engages, for a certain sum, to indemnify the other for loss sustained in his property, which is liable to damage by fire, by the perils of the seas, or otherwise; and is effected on houses or other buildings, ships, goods, &c.

The party who assumes the risk is called *the insurer, assurer,* or *underwriter,* and the owner of the property, *the insured* or *assured.*

The sum paid for insurance, which is a percentage on the property insured, is called *the premium,* and the instrument containing the contract is called *the policy.*

Ex. 1. At 51 cts. per $100, what is the premium on $6275?

Ans. $32,00¼.

2. Effected insurance of £5000 on the ship Noble Ann, of So. Shields, on a voyage to Buenos Ayres, out and home; what did the premium amount to at 2½ p. c.? Ans. £125.

3. A farmer insures his house for $1000, at a premium of 1¼ p. c.; his barn for $3200, at a premium of 1¾ p. c., and his other out-buildings for $500, at a premium of ¾ p. c. What does he pay on the whole? Ans. $72,25.

Duties.

522. These are imposts levied by government, at an establishment called the Custom-House, on certain kinds of goods imported into a country. On the re-exportation of some goods, an allowance is restored to the exporter, called the *drawback.*

The whole weight of goods, as imported in casks, bags, boxes, chests, &c., is called *gross weight.*

From this weight is first deducted a small allowance for waste, called DRAFT, which is as follows:

On a single package, weighing 1 cwt., (112 lb.) ...	1 lb.
above 1 cwt. and under 2 cwt.	2
" 2 " " 3 "	3
" 3 " " 10 "	4
" 10 " " 18 "	7
" 18 and upwards..........................	9

Tare.

523. This is an allowance made, subsequently to draft, for the bag, box, chest, &c. containing the goods; and in computing it on the remainder, less than ½ lb. is not reckoned; more is called 1 lb.

For leakage and breakage 2 p. c. is allowed on the gauge of all merchandise in casks paying duty by the gallon; 10 p. c. on beer, ale, and porter, in bottles, and 5 p. c. on all other bottled liquors. There are other commercial allowances which are not recognised at the Custom-House.

After the deduction of all allowances, the remainder, or real weight of the goods, on which duty is computed, is called *net weight.*

1. What is the duty on 4 boxes of sugar, weighing gross, as follows: 1st. 6 cwt. 1 qr.; 2d. 6 cwt. 3 qrs.; 3d. 7 cwt. 1 qr. 14 lb.; 4th, 7 cwt. 2 qrs. 10 lb.; draft as per table, tare 15 p. c.; the duty being 1¾ cts. per lb.? Ans. $46,36.

2. What is the duty on the above 4 boxes, allowing the same draft, and tare 14 lb. per cwt., at 1¾ d. sterling per lb.?
Ans. £19 17 s. 8¼ d.

Tax.

524. This is a general tribute imposed on the citizens of a country for the support of its government, and is of two kinds—*poll* and *property* tax.

Poll tax is so called because it is levied on the head of every adult male citizen, without regard to property.

Property is taxed at a certain percentage of its amount.

1. Required the tax on an estate worth $17396 at ⅜ p. c.
Ans. $65,23½.

2. Required the tax, at the same rate, on £6397 14 s. 6 d.
Ans. £23 19 s. 10 d. —

General Average.

525. When, in distress, the master of a ship, for the general safety, cuts away her masts, throws part of her cargo overboard, enters a port to repair damages, or is at extraordinary expense to get her afloat when stranded, &c., the loss sustained in consequence is, by a *general average*, equitably proportioned amongst the proprietors of the various interests of said ship, freight, and cargo, which interest must be cleared of all other charges before the average is made.

Property lost by being thrown overboard is called *jettison*, from the French *jeter*, to throw; and this contributes to the general average, otherwise its owner would not share the general loss.

When jettison consists of masts, rigging, spars, &c., a deduction of one-third of the cost of replacing them is made in consequence of the superiority of new articles.

Also from the freight contribution a deduction of ⅓ is usually made (in N. Y. one-half) for seamen's wages.

Ship Clio, New York to Charleston, S. C., grounds on Charleston bar, and lies over to starboard; the master cuts larboard main shrouds and fastenings of deck-load, which being heavy, flies outboard, carrying away starboard bulwarks. The mainmast goes by the board, carrying away mizzen and foretopmast and jib-boom. After clearing wreck, though she does not look very pretty, she behaves well, rights, floats, is towed in by two steamboats, and makes a pretty general average, as follows:

W. H. Harrison's goods started overboard...........	$5000,00
Expense of Steamboats..................................	225,00
Freight lost by jettison..................	75,00
Masts, spars and rigging, $900, less ⅓...............	600,00
Other damage, and premium............................	800,00
Protest, $15. Adjustment of Average, $60..........	75,00
	$6775,00
Agent's commission, 5 p. c................................	338,75
Amount to be made good by average $60...........	$7113,75

Contributory Interests.

Ship, valued at	$35000		
Less damage and premium	1400		$33600,00
Freight, gross amount	$3000, less ⅓............		2000,00
Cargo shipped by J. Phillips.............		$40000	
" " P. Stuyvesant.........		25000	
" " W. Van Twiller......		16450	
" " L. Dow................		15175	
" " W. H. Harrison......		5000	
		$101625	
Amount of shipments................................			101625,00
			$137225,00

Now the whole loss, divided by the whole amount on which it is levied, will give the rate per dollar. This is shown by the proportion $137225 : $1 : : $7113,75 : $,05184004 +. Hence each interest, multiplied by ,05184 +, will give the amoun with which it is charged, thus :

The ship (33600 × ,05184 +) pays.................	$1741,83
" freight (2000 × ,05184 +).......................	103,68
" cargo (101625 × ,05184 +)....................	5268,24
	$7113,75

J. Phillips (40000 × ,05184) pays...	$2073,60
P. Stuyvesant, 25000....................	1296,00
W. Van Twiller, 16450..................	852,77
L. Dow, 15175............................	786,67
W. H. Harrison, 5000..................	259,20
	$5268,24

SECTION XXVI.

EQUATION OF PAYMENTS—AVERAGE OF SALES—PARTNERSHIP—MONEYS—REDUCTION OF CURRENCIES.

Equation of Payments.

526. WHEN several sums become due at different times, we find, by this rule, an equated or mean time for the payment of the whole, so that there may be no loss of interest to the debtor or creditor.

Suppose A owes B $60, due in 4 mo.; $80, due in 6 mo., and $20, due in 8 mo.; what is the equated time for payment of the whole?

The int. of $100 for 1 year is equal to that of $1 for 100 years. Thus, at 6 p. c. the int. of $100 for one year is $6. The int. of $1 is ,06, and (503) the int. of $1 for 100 yrs. is ,06 × 100 = $6.

In like manner, we might show that the int. of $100 for 2 yrs. is equal to that of $2 for 100 yrs., or of $1 for 200 yrs., &c. Hence it is plain that in multiplying the principal by the time of its credit, we have the int. of $1 for the time specified by the product.

We therefore say the int. of

$60 for 4 m. = the int. of $1 for 240 mo.
80 " 6 m. = " 1 " 480 "
20 " 8 m. = " 1 " 160 "

The int. of $160 for *equated time* = int. of $1 for 880 mo. That is, 160 mult. by equated time = 880. Also, as the whole interest is equal to that of $1 for 880 mo., it is plain that $160 will, to gain the same int., require only $\frac{1}{160}$ part of that time; consequently $\frac{880}{160} = \frac{11}{2} = 5\frac{1}{2}$ mo., the equated time required.

Hence the rule: Multiply each payment by the time of its credit. Divide the sum of the amounts by the sum of the payments; the quotient is the equated time.

Average of Sales.

527. To close by one Note, the account of several bills of merchandise bought at various times of credit:

Place the payments under each other in the order in which they become due. Equate the whole from the time the first payment is due, reckoning the equated time forward from the time of the first payment. This is the date when the whole falls due; the Note must, therefore, be dated so as to bring the payment to this date. If drawn payable 4, 6, &c. months after date, the Note must be dated back 4, 6, &c. months accordingly.

MARBLEHEAD, *June* 6, 1854.

JOHN DOE

Bo't of Richard Roe.

Mar. 20,	merchandise	at 8 mos	$1144,00
April 4,	"	4 mos	736,00
" 12,	"	5 mos	900,00
May 10,	"	6 mos	650,00
June 6,	"	6 mos	1566,00
			$4996,00

Time in days, at which the payments become due.

Aug. 4	$736 ×	0	
Sept. 12	900 ×	39	35100
Nov. 10	650 ×	98	63700
Nov. 20	1144 ×	108	123552
Dec. 6	1566 ×	124	194184 d.
by	4996	divide	) 416536 (83 + eq. time.

From Aug. 4, reckoning forward 83 days, we find that the whole amount, $4996, becomes due Oct. 26, 1854. The Note to close this account, will therefore run thus:

$4996. MARBLEHEAD, *June* 26, 1854.

Four months after date, I promise to pay to the order of Richard Roe, Four thousand, nine hundred and ninety-six dollars, without defalcation, value received. JOHN DOE.

Due Oct. 26–29, allowing 3 days, usual grace.

1. A owes B $80, due in 6 mos., $120, due in 8 mos., and $400, due in one year. A, wishing to pay at once, would know the time. Ans. 10$\frac{2}{5}$ mos., or 10 mos. 12 d.

2. A owes B £30 2 s. 6 d., due in 4 mos.; £15 8 s. 4 d., due in 6 mos., and £60 3 s. 4 d., due in 9 mos. A, wishing to pay cash down, B allows him 5 p. c. per an. discount; how much does A pay? Ans. £102 13 s. 1$\frac{1}{4}$ d. + $\frac{8499}{52249}$ qr.

3. Find the int. of £102 13 s. $1\frac{1}{4}$ d. + $\frac{8499}{52249}$, for $7\frac{349}{2537}$ mos., at 5 p. c.? Ans. £3 1 s. $0\frac{1}{2}$ d. + $\frac{43750}{52249}$.

4. Find the discount of £105 14 s. 2 d., for $7\frac{349}{2537}$ mos., at 5 p. c.?

5. A owes B $1000, payable in 8 mos., and pays $250 in 2 mos.; at what time should he pay the remainder?

Ans. 8 mos. from time of first payment.

6. A owes B £198, due in 6 mos., of which he pays $\frac{1}{3}$ in 2 mos., and at the end of 1 mo. from that time he pays $\frac{1}{3}$ of the remainder; when should he make final payment?

Ans. $7\frac{1}{2}$ mos. from time of second payment.

7. A buys goods to the amount of $1000, at 6 months credit. In 3 mos. he pays $330. Wishing to give two Notes for the balance, one at 5 mos., and the other at 9 mos.; required the amount of each Note.

$$\begin{array}{rcl} \$1000 \text{ at 6 mos.} & = & \$6000 \text{ at 1 mo.} \\ \underline{330} \text{ at 3 mos.} & = & \underline{990} \text{ at 1 mo.} \\ 670 \text{ at — mos.} & = & 5010 \text{ at 1 mo.;} \end{array}$$

that is, $\frac{5010}{670}$ average time in months for balance.

$$\frac{5010}{670}\left\{\begin{matrix}\\5\\9\end{matrix}\right\}$$ Times of required Notes, the one greater and the other less than the average time.

As the average time is fractional, we multiply all the three times by 670. Taking the results, which are proportionate, and averaging these, we have

$$5010\left\{\begin{matrix}3350\\6030\end{matrix}\right]\begin{array}{ll}1020 & \text{number corresponding to 5 mos.}\\ \underline{1660} & \text{“ \quad “ \quad 9 mos.}\\ 2680 & \end{array}$$

$$\begin{array}{l} 2680 : 1020 : : 670 : 255 \text{ amount at 5 mos.} \\ \qquad\qquad\qquad \text{hence } 415 \quad \text{“} \quad \text{9 mos.} \end{array}$$

$$\begin{array}{lrl} \text{Proof.} & 330 \text{ @ 3 mos.} = 990 & \\ & 255 \text{ “ 5 “} = 1275 & \text{@ 1 mo.} \\ & \underline{415} \text{ “ 9 “} = \underline{3735} & \\ \text{Total} & 1000 \text{ @ 6 mos.} = 6000 & \end{array}$$

8. A buys goods to the amount of $1200 @ 6 mos., and would give notes for the amount at 3, 5, and 9 mos.

As 6 is an ar. mean between 3 and 9, one part at 3 and one at 9 will equal 2 parts at 6. But as the diff. between 5 and 6 is 1, while that between 6 and 9 is 3, the amounts being inversely as the times, we must have 3 at 5 for 1 at 9, so that the excess in one may equal the deficit or lack in the other. We have then 1 part at 3 mos., 2 at 9 mos., and 3 at 6 mos., making 6 parts of our principal $1200. Hence each part is $200. Then

$$\begin{array}{rl} 200 \text{ @ } 3 \text{ mos.} &= 600 \\ 400 \text{ @ } 9 \text{ mos.} &= 3600 \\ 600 \text{ @ } 5 \text{ mos.} &= 3000 \\ \hline 1200 \text{ @ } 6 \text{ mos.} &= 7200 \end{array}$$

Or thus:

$$6\left\{\begin{array}{c}3\\5\\9\end{array}\right.\quad\begin{array}{l}\cancel{3}\\\cancel{3}\\3+1=\end{array}\quad\begin{array}{c}3\\3\\4\end{array}$$

Here, as we have 10 parts, 1200 ÷ 10 = 120, value of each part.

$$\begin{array}{l} \text{Then } 120 \times 3 = 360 \text{ Note @ } 3 \text{ mos.} \\ \phantom{\text{Then }} 120 \times 3 = 360 \;\;\text{“ @ } 5 \text{ mos.} \\ \phantom{\text{Then }} 120 \times 4 = 480 \;\;\text{“ @ } 9 \text{ mos.} \end{array}$$

$$\begin{array}{lrl} \text{Proof.} & 360 \text{ @ } 3 \text{ mos.} &= 1080 \text{ @ } 1 \text{ mo.} \\ & 360 \text{ @ } 5 \text{ mos.} &= 1800 \text{ @ } 1 \text{ mo.} \\ & 480 \text{ @ } 9 \text{ mos.} &= 4320 \text{ @ } 1 \text{ mo.} \\ \hline & 1200 \text{ @ } 6 \text{ mos.} &= 7200 \text{ @ } 1 \text{ mo.} \end{array}$$

NANTUCKET, July 10, 1854.

THOMAS COFFIN

1854. Bo't of Preserved Fish.

Jan.	4,	Mdse.	at 9 mos.	$1270,25
“	12,	“	“ 4 mos.	560,50
Feb.	14,	“	“ 6 mos.	2000,00
“	22,	“	“ 3 mos.	823,00
May	4,	“	“ 5 mos.	979,75
July	10,	“	“ 8 mos.	1782,50
				$7416,00

What is the date of a Note at 6 months, to close the above account? Ans. April 2, 1854.

Partnership.

528. When several individuals advance a certain sum each, with which they form a capital for trading purposes, agreeing to share equitably the gain or loss resulting therefrom, they take the name of a company or firm.

The sum advanced by each is called his stock in trade, and the gain or loss is shared by each in proportion to his stock, when the stock has all been employed the same length of time. Rule: Divide the whole gain or loss by the whole capital, which will give the gain or loss of one unit of capital. Reduce this to its lowest terms, and multiply each man's stock by the result for his share of the gain or loss. This rule is obtained from the proportion—the whole stock : to each man's : : whole gain or loss : each man's share.

1. A, B, C, and D enter into partnership. A advances \$500; B, \$700; C, \$800, and D, \$1000. In one year they gain \$3500; what is each man's share?

Ratio of gain per unit, $\frac{7}{6}$. Hence we add $\frac{1}{6}$ to each man's stock, and apportion to A, \$583,33$\frac{1}{3}$; B, \$816,66$\frac{2}{3}$; C, \$933,33$\frac{1}{3}$, and D, \$1166,66$\frac{2}{3}$.

2. A, B, and C buy a ship. A pays \$5000; B, \$6000, and C, the remainder, which is the cost of $\frac{1}{5}$ of the ship. The ship is chartered for \$2000; what is the share of each?

Ans. A has \$727,27$\frac{3}{11}$; B, \$872, 72$\frac{8}{11}$, and C, 400.

3. A, B, and C buy a ship. A pays \$4600; B pays for $\frac{1}{3}$, and C pays the remainder, which is $\frac{14}{15}$ of the share of B, together with $\frac{9}{23}$ of that of A. The ship gains \$5000; what is each man's share?

Ans. A, \$1277,77$\frac{7}{9}$; B, \$1666,66$\frac{2}{3}$, and C, \$2055,55$\frac{5}{9}$.

4. A bankrupt owes £16000; his effects amount to £9866 13 s. 4 d.; what does he pay on the pound, and on each of the claims, A, £3085 13 s. 6 d.; B, £5216 11 s. 8 d., and C. £7697 14 s. 10 d.?

Ans. $\frac{37}{60}$, or 12 s. 4 d. on the pound. On the claims: A, £1902 16 s. 7$\frac{3}{4}$ d. $\frac{3}{5}$ qr.; B, £3216 17 s. 10$\frac{1}{4}$ d. $\frac{1}{3}$ qr.; and C, £4746 18 s. 9$\frac{3}{4}$ d. $\frac{1}{15}$ qr.

529. When the stocks are employed for different periods of time: Multiply each man's stock by the time it has been employed, and proceed with the results as in the preceding article.

5. A begins business with \$1000; after 6 months B joins him with \$1500, and in 3 mos. more, C joins with \$1200 They trade for 3 yrs. 6 mos. from the time A begins, and gain \$5000; what is each man's share?

A	\$1000 × 3 yrs. 6 mos., or 3$\frac{1}{2}$ yrs.	3500	Ans. A,	\$1548,67$\frac{29}{113}$
B	1500 × 3 yrs.……………………	4500	B,	1991,15$\frac{5}{113}$
C	1200 × 2 yrs. 9 mos., or 2$\frac{3}{4}$ yrs.	3300	C,	1460,17$\frac{79}{113}$
		11300	Proof,	5000,00

$\frac{5000}{11300} = \frac{50}{113}$, ratio of gain per unit of increased capital:

wherefore each man's increased capital, multiplied by $\frac{50}{113}$, gives his share of gain.

6. Bragg, Oliver, and Richardson commence business with £50000. Bragg advances £25000; Oliver, £16000, and Richardson, £9000. At the end of 6 mos. Bragg withdraws £8000; four mos. after Richardson adds £4000, and 1 yr. 3 mos. from commencement Oliver withdraws £5000. They trade 3 years and gain £25000; what is each man's share?

Bragg,	25000 × $\frac{1}{2}$ yr.	 12500	55000	660	000
"	17000 × $2\frac{1}{2}$ "	 42500			
Rich'n.,	9000 × $\frac{5}{6}$ "	 7500	$37833\frac{1}{3}$	454	000
"	13000 × $2\frac{1}{3}$ "	.... $30333\frac{1}{3}$			
Oliver,	16000 × $1\frac{1}{4}$ "	 20000	44750	537	000
"	11000 × $1\frac{3}{4}$ "	 24750			
				1651	

We here multiply each increased capital by $\frac{12}{1000}$. Then as the results (413) are in the same ratio, we operate with these as we should have done with the capitals, and find

Bragg's share,	£9993	18 s.	$10\frac{1}{4}$ d.	$\frac{565}{1651}$ qr.
Richardson's "	6874	12	5	$\frac{964}{1651}$
Oliver's "	8131	8	$8\frac{1}{2}$	$\frac{122}{1651}$
	£25000	0	0	

7. Reynolds and Elliot begin business with $16000, of which Reynolds advances $\frac{5}{9}$, and Elliot the remainder. After 5 mos. they are joined by Howard, with $12000. They trade during 3 yrs. 1 mo., and gain $13500; what is each man's share?

Ans. Reynolds, $4549,18$\frac{2}{81}$; Elliot, $3639,34$\frac{26}{81}$; and Howard, $5311,47$\frac{33}{81}$.

8. Pitt and Fox commence business with £5000 each. After 8 mos., Pitt advances £5000 more. In one year they are joined by Melbourne, with a capital of £7000. Six months after this they are joined by Grey, with a capital of £6000. At this time Pitt withdraws £2500, and Melbourne puts in £3000 more. They close at the end of 4 years, and find a gain of £15680 16 s. What is the share of each?

Ans. Pitt..........	£3601	18 s.	11 d.	$\frac{17}{1589}$
Fox..........	2368	8	0	$\frac{1056}{1589}$
Melbourne...	7934	2	11	$\frac{1313}{1589}$
Grey..........	1776	6	0	$\frac{792}{1589}$
	£15680	16 s.	0 d.	

Moneys.

530. Money is a general term for coin, paper, &c. used as a circulating medium to represent property, and transfer the same from one person or party to another. It is distinguished into real and imaginary.

Real Money consists of coin, bank notes, or any other tokens which pass current in exchange for property.

Imaginary, or Ideal Moneys, are moneys of account, originally based on coins or weights, and though no longer represented by coins, their value is determined by a known relation to them.

Money is in different countries called by a great variety of names, as cash, specie, currency, effective, &c. That which does not circulate but remains in Banks of Deposit, is called Banco; this, compared with other currency, generally bears a premium called *the agio.* That which circulates by way of Exchange, is called Money of Exchange, (*Moneta di Cambio,* It. or Giro.)

Valuta or Valeur is the value or rate at which moneys are reckoned in commercial transactions.

Reduction of Currencies.

531. Currency is a term often used to denote the money of a country which is in circulation, whether coin or paper.

In the United States the term is sometimes applied to a paper money long discontinued, which on account of too great issue was subject to various rates of depreciation. Hence arose different currencies based upon the proportionate rise in the nominal value of the Pound Sterling and Spanish Dollar. These currencies have still a nominal existence in local domestic traffic, and are as follows:

1. In Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, Connecticut, Virginia, Ohio, Indiana, Kentucky, Tennessee, and Mississippi, the value of the pound currency to that of the pound sterling is as 3 to 4. Hence, as the numbers are in inverse ratio to the values of their principal units, the dollar, being originally valued at 4 s. 6 d. Sterling, is 6 s. of this currency.

Hence also, to reduce this Currency to Sterling, subtract $\frac{1}{4}$. To reduce Sterling to this Currency add $\frac{1}{3}$.

The *real* of Spain, or $12\frac{1}{2}$ ct. piece, being \$$\frac{1}{8}$ or $\frac{1}{8}$ of 72 d.

Currency, is here called *ninepence*, and its half *fourpence halfpenny*.

2. In New York and North Carolina, the ratio is 9 to 16; that is, 9 s. Ster. = 16 s. Cur.; consequently, $9 : 4\frac{1}{2} :: 16$, or $2 : 1 :: 16 : 8$ s., value of the dollar, in currency.

The ratio of this currency to the preceding being as 6 to 8, or 3 to 4, is the same as that of the preceding to Sterling, and is, therefore, reduced to that Currency by subtracting $\frac{1}{4}$. Also, as its ratio to Sterling is duplicate, that is, the square of its ratio to the preceding, thus: $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, it is reduced to Sterling by twice subtracting $\frac{1}{4}$ or by multiplying by $\frac{9}{16}$.

The $12\frac{1}{2}$ ct. piece is here called *a shilling*, because it corresponds with 1 s. Currency, each being $\$\frac{1}{8}$, and its half is called *sixpence*.

3. In New Jersey, Pennsylvania, Delaware, and Maryland, the ratio of Currency to Sterling is as 3 to 5. Hence we have $3 : 4\frac{1}{2} :: 5$, or $2 : 3 :: 5 : 7\frac{1}{2}$, or 7 s. 6 d., value of the dollar in Currency. To reduce to Sterling, multiply by $\frac{3}{5}$. To reduce Sterling to Currency, multiply by $\frac{5}{3}$.

The ratio of 7 s. 6 d. to 8 s. is as 15 to 16; therefore, to reduce this to N. Y. currency, add $\frac{1}{15}$. To reduce N. Y. to this, subtract $\frac{1}{16}$.

Again, 7 s. 6 d. is to 6 s. as 15 to 12, or as 5 to 4; wherefore, to reduce this to Virginia currency, subtract $\frac{1}{5}$. To reduce Virginia currency to this, add $\frac{1}{4}$.

As $\frac{1}{8}$ of 7 s. 6 d., or 90 d., is $11\frac{1}{4}$ d., the $12\frac{1}{2}$ ct. piece is here called *elevenpence*, which is contracted to *levy;* and its half is called *fivepence*, which is contracted to *fip*.

4. In South Carolina and Georgia, the ratio is as 27 to 28: therefore, to reduce Sterling to this add $\frac{1}{27}$. Hence we have 4 s. 6 d. = 54 d. and $54 + \frac{54}{27} = 54 + 2 = 56$ d. = 4 s. 8 d., the value of the dollar in this Currency. Here $12\frac{1}{2}$ cts. is 7 d.

532. Accounts are kept in Canada, Halifax, and other provinces of North America, belonging to England, in Pounds, Shillings, and Pence currency. The ratio is as 9 to 10; that is, £9 Sterling is equal to £10 Currency: therefore, to reduce Sterling to Currency, add $\frac{1}{9}$. To reduce Currency to Sterling, subtract $\frac{1}{10}$.

The dollar, 4 s. 6 d. Sterling, is, therefore, in this Currency 5 s.; but, in army payments, it is reckoned at 4 s. 8 d. To reduce Canada currency to Federal money, considering the dollar to be worth 5 s. or £$\frac{1}{4}$, multiply the pounds by 4.

Thus, to reduce £100 16 s. currency to Federal money, we say £$100\frac{4}{5} \times 4 =$ \$$403\frac{1}{5}$, or \$403,20. To reduce this again to currency, divide by 4. Thus:

$403,2 \div 4 =$ £100,8; or £100 16 s. *Note.* Double the tenths for shillings.

Reckoning the dollar at 4 s. 8 d. = £$\frac{7}{30}$, multiply by $\frac{30}{7}$. Thus the above sum in Federal money becomes

$$£100,8 \times \frac{30}{7} = 14,4 \times 30 = 144 \times 3 = \$432.$$

General Rule for Reduction of Currencies:

533. To reduce one currency to another,—that is, to express the same value by two different numbers,—implies a difference in the magnitude or value of their principal units; and (288) the numbers are to each other in inverse ratio to these values: hence *we refer the principal unit of each currency to the same known standard, and thus establish their ratio. We then multiply the given number by the reciprocal of its ratio to the required number; that is, by the ratio of the value of the principal unit of the given to that of the required number,* (219.)

Examples.

1. Reduce 50000 soldi piccoli of Venice to grani of Palermo: $36\frac{1}{2}$ grani or $34\frac{1}{4}$ soldi piccoli being equal to 1 lira, *fuori banco, (out of bank.)*

$36\frac{1}{2} = \frac{73}{2} = \frac{146}{4}$ and $34\frac{1}{4} = \frac{137}{4}$. Hence (189 and 288) the ratio of the value of the soldo piccolo to that of the grano is $\frac{146}{137}$, we therefore say (219) $\frac{50000 \times 146}{137} = \frac{8300000}{137} =$ $60583\frac{129}{137}$, or 60584 grani, nearly.

2. Reduce £8415 Pennsylvania to New York currency. Answer, £8976.

3. Reduce £2618 15 s. Canada currency to Federal money, the dollar being 4 s. 8 d. Answer, \$$11223,21\frac{3}{7}$.

4. What is the worth of £2618 15 s. Canada currency, the pound sterling being \$4,87? Answer, \$$11477,98\frac{1}{8}$.

SECTION XXVII.

EXCHANGE—CHAIN RULE—COMPOUND INTEREST—ANNUITIES—INCREASE OF POPULATION.

Exchange.

534. This is the method of discharging debts, paying and receiving money, &c., by remitting bills instead of cash, coin, or bullion: (bullion is gold or silver uncoined.)

If the remittance is made to a place in the same country, the transfer of claims is called *Inland Exchange.*

535. When a sum of money of one kind or country is immediately bartered for its equivalent in that of another, the operation is called a Common or Dry Exchange, (Cambio Comune, It.;) but when paid in one place for a bill or order for its equivalent in the money of another, the transaction is called a Real Exchange, (Cambio Reale:) also Foreign Exchange.

536. A Bill of Exchange is a written order, addressed to an individual, firm, or company, at a distance, for the payment, at an appointed time, of the sum specified in the bill, to the person or party in whose favour the bill is drawn, or to the order of such person or party. As a precaution against accident, it is usual to draw three copies of a foreign bill, and send them by different posts. They are called First, Second, and Third of Exchange; and when any one of them is paid, the rest become void.

Form of a Bill of Exchange.

First, PHILADELPHIA, *Mar.* 2, 1854.

Exchange £6000 sterling.

At sixty days' sight of this, my first of Exchange, (second and third of the same tenor and date unpaid,) pay to George Heath or order, Six Thousand Pounds Sterling, with or without further advice from me. LYON J. LEVY.

CHARLES CHEERYBLE, merchant, London.

Suppose that Lyon J. Levy sells this bill to Thomas P. Cope, then,

1. Lyon J. Levy is called the *drawer*, or *maker*, and *seller* of the bill.

2. Charles Cheeryble is the *drawee*, and when he accepts the bill, by writing his name across it with the word *accepted*, and thus engages to pay it when due, he is called *the acceptor.*

3. Thomas P. Cope, who gives value for the bill, is called the *buyer*, *taker*, and *remitter*.

4. George Heath is the *payee*, who may, by endorsement, pass it to another.

Mercantile payments are mostly made in Bills of Exchange, which pass, till due, from hand to hand, like any other circulating medium: and he who, at any time, has a bill in his possession, is called the *holder*.

When the holder disposes of the bill, he writes his name on the back of it, which is called *endorsing;* and the *payee* should be the first endorser.

If the bill be endorsed in favour of any particular person, it is called a *special endorsement*, and the person to whom it is thus made payable is called the *endorsee*, who must also endorse the bill if he negotiates it.

Any person may endorse a bill, and every endorser (as well as the acceptor or payee) is a security for the bill, and may be sued for payment.

The term or time of a bill is conventional or regulated by custom. Some bills are drawn at sight; some at so many days or months after sight or after date; and some at *usance;* that is, at or for the *usual term.*

A certain number of days (in the United States and British dominions, three) are allowed the acceptor, for payment of the bill after its term has expired: these are called *days of grace.*

In reckoning when a bill, payable after date, becomes due, the day on which it is dated is not included; and, if payable after sight, the day of presentment is not included. When the term is expressed in months, calendar months are understood; and when a month is longer than the succeeding, it is a rule not to go, in the computation, into a third month. Thus, if a bill be dated the 28th, 29th, 30th, or 31st of January, and payable one month after date, the term equally expires on the last day of February, to which the days of grace must, of course, be added; and therefore the bill becomes due on the third of March.

Inland Bill or Draft.

$690. PORTLAND, March 29, 1854.

Sixty days after date, pay to the order of Charles Horton, Six Hundred and Ninety Dollars, value received, and charge the same to the account of RUFUS D. BEAN.

To ISAAC WINSLOW, merchant, Philadelphia.

For payment or acceptance, a bill should be presented during the usual hours of business; that is, from 9 A.M. till 6 P.M. When either has been refused, the holder should give immediate notice to the parties he intends to hold responsible for its payment. This, in foreign exchange, must be accompanied by a *protest,* which is an instrument drawn by a notary, showing that acceptance or payment has been demanded and refused, and that the holder intends to recover any damages which he may sustain in consequence.

Chain Rule.

537. This rule is considered indispensable in the higher operations of exchange, and is, therefore, called the *Rule of Exchange.* It is, from its nature, also called *Conjoint Proportion* and *Rule of Reduction:* because it is applied to solve a complicated reduction of currencies, and effects, by one operation, similar to a Compound Proportion, what would otherwise require several statements in the Rule of Three.

Learners, not being generally acquainted with foreign moneys, to the reduction of which this rule is generally applied, may more readily understand it as illustrated by the following more familiar objects:

1. If 5 lb. of tea be worth 25 lb. of coffee, 7 lb. of coffee worth 16 lb. of sugar, and 20 lb. of sugar worth 8 lb. of cocoa, what quantity of cocoa should we have for 27 lb. of tea?

Put $A = 1$ lb. of tea, $B = 1$ lb. of coffee, $C = 1$ lb. of sugar, and $D = 1$ lb. of cocoa. Then (288) the ratio $\frac{A}{B} = \frac{24}{5}$, $\frac{B}{C} = \frac{16}{7}$, $\frac{C}{D} = \frac{8}{20} = \frac{2}{5}$.

Hence, the ratio of A to D—that is, of 1 lb. of tea to 1 lb. of cocoa, which (214) is compounded of the ratios A to B, B to C, and C to D—is $\frac{24}{5} \times \frac{16}{7} \times \frac{2}{5} = \frac{768}{175}$; consequently, (219) $\frac{27}{1} \times \frac{768}{175} = 118\frac{86}{175}$ lb., the required quantity of cocoa.

From this example, we easily deduce the following

General Rule.

Draw a horizontal line, and place the Term of Demand, which is here 27, on the left, above the line as a numerator, and its homogeneous term under the line, one place towards the right, as a denominator; then the equivalent of this, as shown by the sign // one place farther to the right, as a numerator; then the term of the same kind one place to the right, as a denominator, and so on through all the terms; the last of which, having no homogeneous term, is called The Odd Term, and is, as well as the term of demand, a numerator. The result, after cancelling, is found as in multiplication of fractions. Thus, for the above example, we have

$$\frac{27 \times 24 \times 16 \times 8}{5/\!/ \quad 7/\!/ \quad 20/\!/} = 118\tfrac{86}{175} \text{ lb. of cocoa.}$$

Or the numbers may be arranged in two columns.

First, place the Term of Demand on the right, then that which is of the same kind first on the left, one line lower, and opposite to this its equivalent, with the sign = between them; continue thus to alternate the terms, as has been shown above, the last of which, or Odd Term, will be the last of the right-hand column. Then, as the numbers on the right are the numerators, and those on the left the denominators, after cancelling as much as practicable, the result is found by dividing the product of the numerators by that of the denominators, as in multiplication of fractions.

The above example will stand thus:

27 lb. tea,
5 lb. tea = 24 lb. coffee,
7 lb. coffee = 16 lb. sugar,
5 lb. sugar = 2 lb. cocoa.

Reduced, gives $118\frac{86}{175}$, as above.

2. Suppose N, of the United States, would remit $5000 to Paris, and that the direct Exchange is 5 Francs 40 centimes per dollar, but he wishes to send it through London and Amsterdam; which is most advantageous, the direct or indirect Remittance, the quotation of the Course of Exchange being as follows?—

United States on London, 52 d. Sterling per Dollar.

London on Amsterdam, 10 Florins 10 Stivers per Pound Sterling.

Amsterdam on Paris, 60 d. Flemish per Ecu of 3 Francs.

5000 Dollars.
1 Dollar = 52 Pence Sterling.
240 Pence = 1 Pound Sterling.
1 Pound St. = 10½ Florins.
1 Florin = 40 Pence Flemish.
60 d. Flem. = 3 Francs.

$$\frac{5000 \times 52 \times 10{,}5 \times 40 \times 3}{240 \times 60} = 22750 \text{ Francs.}$$

Proof. 22750 Francs.
3 Francs = 60 Pence Flemish.
40 Pence Fl. = 1 Florin.
10½ Florins = 1 Pound Sterling.
1 Pound St. = 240 Pence Sterling.
52 Pence St. = 1 Dollar.

$$\frac{22750 \times 60 \times 240}{3 \times 40 \times 10{,}5 \times 52} = 5000 \text{ Dollars.}$$

Thus we see that by the indirect Exchange, the Remitter will, for $5000, receive only 22750 Francs, which sum gives the rate of the dollar 4 Francs, 55 Centimes; whereas, by the direct Exchange, he will, at 5 Francs 40 Centimes per dollar, receive 27000 Francs: the latter is, therefore, the most advantageous.

On the contrary, to the Drawer, the indirect Exchange would, in the same proportion, be most advantageous, as will be fully explained hereafter.

538. The Par of Exchange is *Intrinsic* or *Commercial.* Intrinsic Par is the comparative value of the Moneys of two countries with respect to weight and fineness. Commercial Par is their comparative value according to weight, fineness, and the market prices of the metals. Thus two sums of different countries are *intrinsically* at par *when each contains*, and *commercially* at par *when each can purchase*, the same quantity of pure metal of the same kind.

The intrinsic Par between London and the principal trading places of Europe is found by comparing gold with gold, and silver with silver, either from the Mint Regulations of each place or from Assays.

539. In computing the Par of Exchange by the Chain Rule, the *certain money* should be the leading term, or first consequent, and the uncertain money the term of demand, or last consequent, as in the following examples:

London and Amsterdam.—Gold.

1. Required the Par of Exchange between London and Amsterdam resulting from the 10 Florin Piece, weighing 6,729 Grammes $\frac{9}{10}$ fine, and the British Sovereign, or 20 Shilling Piece? Here the latter is the *certain*, and the former the *uncertain* money.

1 Pound Sterling,
1 Pound Sterling = 240 Pence,
934½ Pence = 1 Ounce Standard,
12 Ounces Standard = 11 Ounces fine Gold.
1 Ounce = 480 Grains,
15434 Grains = 1000 Grammes,
6,729 Grammes × $\frac{9}{10}$ = 10 Florins.

Result, 12 Florins 9 cents for £1 Sterling, or 40 Shillings 3 Pence Flemish.

But from Tables in which the quantity of pure metal in each coin is given, the work is performed more readily; thus, if we find that the 10 Florin Piece contains 93,46 Grains, and the Sovereign 113 Grains of fine Gold, say

9,346 : 113 : : 1 Florin : 12,09, as above.

London and Amsterdam.—Silver.

2. Required the Par of Exchange between London and Amsterdam, resulting from a comparison of the new Silver Florin, which contains 148,38 grains of fine silver, and the Pound Sterling, which, in silver, contains 1718,7 grains; hence,

$$\overset{\text{Gr.}}{148,38} : \overset{\text{Gr}}{1718,7} :: \overset{\text{Fl.}}{1} : \overset{\text{Fl.}}{11,58}.$$

Thus the Par in silver is 11 Florins 58 cents for the Pound Sterling, which differs about 4¼ per cent. from the gold Par.

London and Hamburgh.—Gold.

3. Required the Par of Exchange from the Hamburg Ducat containing 53 grains of pure Gold, and the Sovereign containing 113 grains, supposing that the Ducat equals 16 Shillings Flemish.

$$\overset{\text{Gr.}}{53} : \overset{\text{Gr.}}{113} :: \overset{\text{Sh.}}{16} : \overset{\text{Sh.}}{34}\ \overset{\text{d.}}{1,35}.$$

Thus the gold Par is 34 Shillings 1,35 Pence Flemish per £1 Sterling; but from the fluctuating value of the Ducat, the Par cannot be considered permanent; by taking the average price, for 4 years, of the Ducat, which is 6¼ Marks, the Par would be 35 Shillings 6 Pence per Pound Sterling.

London and Hamburgh.—Silver.

4. The Cologne Mark of fine Silver is generally worth $27\frac{5}{8}$ Marks Banco, and 60 Marks Cologne weight are equal to 451 ounces Troy; what is the Par between London and Hamburgh Banco, in Silver?

1 Pound Sterling,
1 Pound Sterling = 240 Pence,
62 Pence = 1 Ounce Standard,
40 Ounces Standard = 37 Ounces fine Silver,
451 Ounces = 60 Cologne Marks,
1 Cologne Mark fine = $27\frac{5}{8}$ Marks Banco,
3 Marks Banco = 8 Shillings Flemish Banco.

Reduced, gives 35 s. 1 d. Flemish Banco, per £ Sterling.

But on account of the fluctuations in the price of the Ducat and of the Mark of fine Silver, no permanent Par can be determined between London and Hamburgh.

London and Paris.—Gold.

5. Required the Par of Exchange between London and Paris, the Kilogramme of gold weighing 15434 Eng. grains, being valued at 3444 Francs, 44 Cen. 444.

1 Pound Sterling,
1 Pound Sterling = 240 Pence,
$934\frac{1}{2}$ Pence = 1 Ounce Standard Gold,
12 Oz. Standard = 11 Ounces fine Gold,
1 Ounce = 480 Grains,
15434 Grains = 1 Kilogramme,
1 Kilogramme = 3444 Francs 44 Cen. 444.

Reduced, gives the Pound Sterling equal to 25 Francs, 22 Centimes.

Suppose that by Assay, the Napoleon or 20 Franc Piece is found to contain $89\frac{1}{2}$ grains of pure gold, then

$$\overset{\text{Gr.}}{89{,}5} : \overset{\text{Gr.}}{113} :: \overset{\text{Fr.}}{20} : \overset{\text{Fr.}}{25}\ \overset{\text{Cen.}}{25}.$$

London and Paris.—Silver.

6. Required the Par of Exchange between England and France, at 62 d. Sterling per ounce of Standard Silver $\frac{37}{40}$ fine; the Kilogramme of silver being valued at 222 Francs, 22 Cen. 222.

1 Pound Sterling,
1 Pound Sterling = 240 Pence,
62 Pence = 1 Ounce Standard,
40 Oz. Standard = 37 Ounces fine,

1 Ounce fine = 480 Grains,
15434 Grains = 1 Kilogramme,
1 Kilogramme = 222 Francs 22 Cen. 222.

Reduced, gives the Pound Sterling equal to 24 Francs, 73 Cen.

London and Genoa.—Gold.

7. The gold Genovina contains 357,7 English grains of fine gold, and it passes for 96 Lire fuori Banco; what is the Par between London and Genoa in gold?

1 Pezza,
1 Pezza = 5¾ Lire,
96 Lire = 1 Genovina,
1 Genovina = 357,7 Grains of fine Gold,
480 Grains = 1 Ounce,
11 Oz. fine Gold = 12 Ounces Standard,
1 Oz. Standard = 934½ Pence Sterling.

Reduced, gives 45½ d. Sterling per Pezza.

London and Genoa.—Silver.

8. The silver Scudo contains 457½ English grains of fine silver, and it passes for 8 Lire fuori Banco; what is the Par between London and Genoa in silver?

1 Pezza,
1 Pezza = 5¾ Lire,
8 Lire = 457½ Grains of fine Silver,
480 Grains = 1 Ounce
37 Oz. fine Silver = 40 Ounces Standard.
1 Ounce = 62 Pence Sterling.

Reduced, gives 45,92 d. Sterling per Pezza.

London and Leghorn.—Gold.

9. The Sequin contains 53,6 Eng. grains of fine gold, and it passes for 13⅓ Lire Moneta Buona; what is the Par between London and Leghorn in gold?

1 Pezza,
1 Pezza = 5¾ Lire Moneta Buona,
13⅓ Lire Mon. Buona = 1 Sequin,
1 Sequin = 53,6 Grains of fine Gold,
480 Grains = 1 Ounce,
11 Oz. fine Gold = 12 Ounces Standard,
1 Oz. Standard = 934½ Pence Sterling.

Reduced, gives 49,09 d. Sterling per Pezza.

London and Leghorn.—Silver.

10. The Silver Scudo or Leopoldone contains 384 Eng. grains of fine Silver, and it passes for $6\frac{1}{3}$ Lire Moneta Buona; what is the Par between London and Leghorn, in silver?

1 Pezza,
1 Pezza = $5\frac{3}{4}$ Lire Mon. Buona,
$6\frac{1}{3}$ Lire Mon. Buona = 1 Scudo,
1 Scudo = 384 Grains of fine Silver,
480 Grains = 1 Ounce,
37 Oz. fine Silver = 40 Ounces Standard,
1 Oz. Standard = 62 Pence Sterling.

Reduced, gives 48,68 d. per Pezza.

London and Spain.—Gold.

11. The Quadruple of 1772 contains 372 Eng. grains of fine Gold, and it passes for 320 Reals Vellon; what is the Par between London and Spain, in gold?

1 Dollar of Plate,
1 Dollar of Plate = 8 Reals of Plate,
17 Reals of Plate = 32 Reals Vellon,
320 Reals Vellon = 1 Quadruple,
1 Quadruple = 372 Grains of fine Gold,
480 Grains = 1 Ounce,
11 Oz. fine Gold = 12 Ounces Standard,
1 Oz. Standard = $934\frac{1}{2}$ Pence Sterling.

Reduced, gives 37,18 d. Sterling per Dollar of Plate.

London and Spain.—Silver.

12. The Dollar contains 371 Eng. grains of fine Silver, and it passes for 20 Reals Vellon; what is the Par between London and Spain, in silver?

1 Dollar of Plate,
1 Dollar of Plate = 8 Reals of Plate,
17 Reals of Plate = 32 Reals Vellon,
20 Reals Vellon = 1 Hard Dollar.
1 Hard Dollar = 371 Grains of fine Silver,
480 Grains = 1 Ounce,
37 Oz. fine Silver = 40 Ounces Standard,
1 Oz. Standard = 62 Pence Sterling.

Reduced, gives 39 d. Sterling per Dollar of Plate.

London and Lisbon.—Gold.

13. The piece of 6400 Rees contains 203 Eng. grains of fine Gold; what is the Par between London and Lisbon, in gold?

1000 Rees,
6400 Rees = 203 Grains of fine Gold,
480 Grains = 1 Ounce,
11 Oz. fine Gold = 12 Oz. Standard,
1 Oz. Standard = $934\frac{1}{2}$ Pence Sterling.

Reduced, gives 67,36 d. Sterling per Milree.

According to the above methods, the other Pars, contained in the following Table, have been computed.

Table 1.

A Table of the Par of Exchange between England and the principal places in Lloyd's List, computed from the intrinsic value of their coins, by comparing gold with gold, and silver with silver, according to their Mint Regulations, and to Assays made at the London and Paris Mints, valuing English gold at £3 17 s. $10\frac{1}{2}$ d. per Oz. Standard, and English silver at 5 s. 2 d. per Oz. Standard.

	GOLD.		SILVER.		EXPLANATIONS.
	Mint Regulations.	Assays.	Mint Regulations.	Assays.	
Amsterdam, in Flemish.....	40 3	......	38,6	......	Shilgs.and pence Flem.per £1stg.
Ditto, in florins	12 09	......	11,58	......	Florins and cents per £1 stg.
Hamburgh.....	34 3,5	34 1,5	35 1	35 1,3	Shilgs.and pence Flem.per £1stg.
Paris............	25 22	25 26	24 75	24 91	Francs and centimes per £1 stg.
Madrid...........	37,3	37,2	39,2	39 0	Pence sterling per piastre.
Lisbon...........	67,4	67,5	....	......	Pence sterling per milree.
Leghorn..........	49,1	49,0	46,46	46 5	Pence sterling per pezza.
Genoa............	45,5	45,5	45,92	45,92	Pence stg. per pezza fuori banco.
Naples...........	41,2	......	41,2	......	Pence sterling per ducat.
— otherwise.	582	......	582	......	Grains per £1 stg.
Venice...........	46,3	46,0	47,5	49,0	Lire piccole per £1 stg.
— otherwise.	23,44	23	24,30	25,07	Italian livres per £1 stg.

It has already been shown that when the quantity of pure metal in the moneys of Exchange is known, the Par is found by division; and it may, in the same manner also, be found from the Table of Moneys of Account, when the moneys of exchange are of the same denomination. Thus, to find the Par between France and England, the Franc is worth 9,7 Pence in Silver, and 9,52 Pence in Gold; and therefore 240, divided by these sums respectively, will give the Par in Gold and Silver, as before.

540. Gold and silver in their pure state are considered too flexible for coin; hence they are mixed with a certain proportion of harder metal, called *alloy*. The alloy of silver is mostly

copper, and that of gold both silver and copper; but in the computations of coins, the alloy is never reckoned of any value.

541. In regular governments a standard for the quantity of pure metal in coins is fixed by law. In the United States and in England the legal standard for gold is $\frac{22}{24}$ or $\frac{11}{12}$; that is, eleven parts of pure metal and one of alloy. The fineness of gold is generally estimated in *carats;* the whole weight being divided into 24 equal parts, or carats. Hence the United States and English standard gold is said to be 22 carats fine; that is, 22 parts of pure gold and 2 of alloy. The carat is divided into 4 equal parts, called *grains.*

542. The English standard for silver is $\frac{222}{240}$ or $\frac{37}{40}$; that is, 11 oz. 2 dwt. of pure silver and 18 dwt. of alloy make 1 lb. Troy.

In England 1 lb. Troy of standard gold is coined into 44½ guineas, and 1 lb. of standard silver into 66 shillings; hence the Mint price of standard gold is £3 17 s. 10½ d. per oz., and that of standard silver, 66 d. per oz. Before the year 1816, silver was coined at the rate of 62 d. per oz., and this is still reckoned the standard price in the valuation of foreign silver coins.

543. In 1790 the United States Government established a Mint, and ordered money to be coined in gold, silver, and copper, according to the following denominations and values, viz.:

Eagles.—Each to be of the value of 10 dollars or units, and to contain 247¼ grains of pure, or 270 grains of standard, gold, the standard being 22 carats fine, or $\frac{11}{12}$. Its intrinsic value in English gold is, therefore, £2 3 s. 8 d., nearly: half-eagles and quarter-eagles to be in the same proportion.

Dollars, or *Units.*—Each to be of the value of a Spanish milled dollar, and to contain 371¼ grains of pure, or 416 grains of standard, silver, the standard being $\frac{1485}{1664}$ fine, or 10 oz. 14 dwt. nearly. Its intrinsic value in English silver is, therefore, 4 s. 3¾ d., nearly: half-dollars, quarter-dollars, dimes, or tenths of dollars, and half-dimes in the same proportion.

Cents.—Each to be of the value of the one-hundredth part of a dollar, and to contain 208 grains of copper: half-cents in the same proportion.

544. The following coins are by Act of Congress rendered current in the United States:

The gold coins of Great Britain, Portugal, and Brazil, not less than 22 carrats fine, at 94,8 cents per dwt. Those of France, $\frac{9}{10}$ fine, at 93,1 cents, and those of Spain, Mexico, and Columbia, 20 carats, $3\frac{7}{16}$ grains, at 89,9 cents per dwt.

The silver dollars of Mexico, Peru, Chili, and Central America, of not less weight than 415 grains each, and those restamped in Brazil, of like weight; not less than 10 oz. 15 dwt. in the Troy pound of standard silver, at $1,00 each.

The 5-franc piece of France, of not less fineness than 10 oz. 16 dwt. in the Troy pound of standard silver, and weighing not less than 384 grains each, at 93 cents.

545. The intrinsic worth of the Spanish *peso duro* (hard dollar) universally circulated under the name of the Spanish dollar, as well as of the dollar of the United States, considered of equal value, is, according to the Cambist, 51,79 d., or 4 s. 3,79 d. sterling. The *nominal par*, however, is 4 s. 6 d., or £$\frac{9}{40}$ sterling. Wherefore, at this par,

To reduce Federal money to Sterling, multiply by $\frac{9}{40}$.

To reduce Sterling money to Federal, multiply by $\frac{40}{9}$.

Table 2.

The Relative Value of Gold and Silver in the Principal Trading Places of the World; computed from the proportional Quantity of Pure Metal in their principal Coins, and the Legal Current Price of those Coins respectively.

(From Kelley's Universal Cambist.)

	By Mint Regulations.	By Assays.	Names of Coins from which the Proportions are taken.
England:			
By old coinage	15,2096 to 1	Proved correct by the trials of the Pix	Per guinea and old shilling.
By new coinage	14,2878 to 1		Per sovereign and new shilling.
Amsterdam	15,8735 to 1		Per ten-guilder piece, decreed in 1816, and silver florin of the same date.
Hamburg	15 to 1, nearly	14,83 to 1	Per ducato, reckoned at 6 marks banco, and rix-dollar.
Paris	15,5 to 1	15,5 to 1	Per 20-franc piece, and 5-franc piece.
Madrid	16 to 1	15,85 16,46 to 1	Per doubloon and dollar of different coinages.
Lisbon	13,56 to 1	13,83 to 1	Per joanese and new silver crusado.
Leghorn	14,65 to 1	14,32 to 1	Per ruspono and francescone.
Genoa	15,34 to 1	15,35 to 1	Per genovina and scudo.
Naples	15,21 to 1		Per oncetta and ducato. (Coinage of 1818.)
Venice	15 to 1, nearly	14,35 to 1	Per sequin and ducat.
Petersburg	15 to 1, nearly	15,25 to 1	Per ducat and ruble.
United States	15 to 1	15,94 to 1	Per eagle and dollar.
Bengal	14,857 to 1	14,827 to 1	Per gold mohur and Sicca rupee.
Madras	13,872 to 1	13,857 to 1	Per star pagoda and current rupee.
Bombay	15 to 1	15 to 1	Per gold rupee and silver rupee.
China	14,25 to 1		Per tale of gold and the average price of Spanish dollars.

546. The foregoing Table may be computed by the Chain Rule, in the following manner:

Required the relative proportion between Gold and Silver in the English Coins, according to the Mint Regulations both of the old and new system.

The question is, to compare the value of any certain quantity, suppose an ounce of pure Gold, with an ounce of pure Silver, at the Mint price:

Old System.	*New System.*
1 oz. pure Gold.	1 oz. pure Gold.
11 oz. pure = 12 oz. Standard.	11 oz. Pure = 12 oz. Standard.
1 oz. Standard = 934$\frac{1}{2}$ Pence.	1 oz. Standard = 934$\frac{1}{2}$ Pence.
62 Pence = 1 oz. Standard Silver.	66 Pence = 1 oz. Standard Silver.
40 oz. Standard = 37 oz. Pure.	40 oz. Standard = 37 oz. Pure.
Reduced gives 15$\frac{2859}{13640}$. Thus Gold, decimally expressed, is to Silver as 15,2096 to 1.	Reduced gives 14$\frac{1393}{4840}$. Thus Gold, decimally expressed, is to Silver as 14,2878 to 1.

The operation is more simple when the rate of fineness of both metals is expressed in the same manner. Thus, in the Coinage of France, the 20-Franc Piece in Gold weighs 6,5161 Grammes in pure Gold, and 20 Francs in Silver weigh 100 Grammes: hence, the latter divided by the former will give 15,5, as in the Table.

When the fineness differs in expression, the comparison may be made from the quantity of pure Gold and of pure Silver in any particular sum, found either by the Mint Regulations or by Assays. Thus the English Sovereign contains 113 Grains of fine Gold; and 20 Shillings, (new coinage,) 1614,54 Grains of fine Silver: hence, the latter divided by the former will give the relative value very nearly as above.

Moneys of Account.—Table 3.

547. In the following Table of Moneys of Account, it may be observed that some of these moneys are real coins, the value of which may be computed from the Mint Regulations or from Assays; but when they are imaginary moneys, which is generally the case, their value must be found by their established proportion to real coins.

Table 3 contains the Value of the Moneys of Account of different Places, (expressed in Pence Sterling and Decimals of Pence,) according to the Mint Price both of Gold and Silver in England; that is, £3 17 s. 10$\frac{1}{2}$ d. per Ounce Standard for Gold, and 5 s. 2 d. per Ounce Standard for Silver.

		Value in Silver.	Value in Gold.
		d.	d.
Aix-la-Chapelle	Rixdollar current	31,40	31,43
Alicant	Libra, or Peso	39,40	37,38
Amsterdam	Rixdollar	52,54	variable.*
“	Florin (Old)	21,	“
“	Florin (New)	20,72	“
“	Pound Flemish	124,32	“
Antwerp	Pound Flemish (Money of Exchange)	123,25	123,87
“	Florin (Money of Exchange)	20,54	20,64
“	Pound Flemish current	105,65	106,18
“	Florin current	17,60	17,70
Arragon	Libra Jaquesa	49,25	46,75
Augsburg	Florin Giro, or money of exchange	32,	31,83
“	Florin current	25,20	25,07
Barcelona	Libra Catalan	28,14	26,70
Basil	Rixdollar, or Ecu of exchange	47,27	47,
“	Rixdollar current	42,45	42,20
Bergamo	Scudo of 7 Lire	35,67	36,50
Berlin	Pound Banco	47,25	variable.
“	Rixdollar current	36,	“
“	Rixdollar in Fredericks	——†	39,68
Berne	Ecu of 3 Livres	42,64	42,90
“	Crown of 25 Batzen	35,53	35,75
Bologna	Lira corrente	10,86	10,62
“	Lira, money of exchange	11,12	10,89
Bolsano	Florin Giro, or money of exchange	33,26	33,08
“	Florin moneta lunga, or currency	25,20	25,06
Bremen	Rixdollar current	37,80	variable.
“	Rixdollars in Carl d'ors	——	39,68
Canary Islands	Real current	3,95	3,66
Cassel	Rixdollar current	37,80	variable.
Cologne	Rixdollar specie of 80 Albuses	31,38	“
“	Rixdollar current of 78 Albuses	30,60	“
Constantinople	Piastre, or Dollar, 1819	9,45	uncertain.
Dantzic	Gulden or Florin	9,	9,
Denmark	Rixdollar specie	54,72	——
“	Rixdollar Sundish specie	53,21	——
“	Rixdollar Crown money	48,37	——
“	Rixdollar Danish currency	44,27	44,88
“	Rixdollar Holstein currency	43,78	44,16
England	Pound Sterling	240,	240,
Florence	Lira	8,12	8,53
“	Ducat, or Crown current	56,84	59,71
“	Scudo d'oro, or Gold Crown	——	63,97
France	Livre Tournois	9,58	9,40
“	Franc (new system)	9,70	9,52
Frankfort	Rixdollar, Convention money	37,80	37,65
“	Rixdollar Muntze, or in small coins	31,50	——
Geneva	Livre current	16,13	16,93
“	Florin	4,60	4,84
Genoa	Lira fuori Banco	7,99	7,83
“	Pezza. or Dollar of exchange	45,92	45,50
“	Scudo di cambio, or Crown of exchange	36,75	36,02
“	Scudo d'oro marche	85,49	83,77
Germany	Rixdollar current	37,80	variable.
“	Rixdollar specie	50,40	“
“	Florin of the Empire	25,20	“
“	Rixdollar Muntze	31,50	“
“	Florin Muntze	21,	“
Hamburg	Mark Banco (at a medium)	18,22	“
“	Pound, Flemish Banco	136,65	“
“	Mark current	14,82	“
“	Pound Flemish current	111,15	“

* In the places marked *variable*, the price of the coins is not fixed; and, therefore, the intrinsic value in gold of the moneys of account cannot be ascertained for any length of time.

† Where the columns are marked with a dash, it is to be understood that there is no coin in the metal of that column by which the moneys of account can be computed.

Table 3, (continued.)

		Value in Silver.	Value in Gold.
		d.	d.
Hanover	Rixdollar, in cash	42,	42,26
“	Rixdollar, gold value	39,	39,24
Ireland	Pound, Irish	221,54	221,54
Konigsberg	Gulden, or Florin	12,	variable.
Leghorn	Pezza of 8 Reals	46,25	49,16
“	Lira moneta buona	8,13	8,55
“	Lira moneta lunga	7,79	8,19
Leipsic	Rixdollar, Convention money	37,80	variable.
“	Rixdollar in Louis d'ors or Fredericks	——	39,68
Lucca	Lira	7,40	7,77
“	Scudo d'oro	55,50	58,27
“	Scudo corrente	51,80	54,39
Malta	Scudo, or Crown	21,32	23,34
Milan	Lira Imperiale	10,41	10,53
“	Lira corrente	7,45	7,30
“	Scudo Imperiale	60,90	61,60
“	Scudo corrente	42,32	42,78
Modena	Lira	3,72	3,15
Munich	Gulden, or Florin	21,	21,28
Nancy	Livre (money of Lorraine)	7,38	7,26
Naples	Ducato di regno	41,20	uncertain.
Navarre	Real	4,90	4,67
“	Libra	8,20	7,79
Neufchatel	Livre Tournois	13,63	13,40
“	Livre foible	5,45	5,36
Novi	Scudo d'oro marche	85,49	83,77
Parma	Lira	2,35	2,30
Persia	Toman of 100 Mamoodis	287,60	——
Poland	Gulden or Florin	6,03	6,27
Portugal	Milree	uncertain.	67,34
“	Old Crusado	“	26,94
Prague	(See Vienna)		
Riga	Rixdollar Alberts	52,54	variable.
“	Rixdollar currency, (agio 40 per cent)	37,53	“
Rome	Scudo, or Crown	52,05	51,63
“	Scudo di Stampa d'oro	79,37	78,73
Russia	Ruble	variable.	
St. Gall	Florin, money of exchange	27,44	variable.
“	Florin current	22,76	“
St. Remo	Lira	8,46	8,90
Sardinia	Lira	18,21	18,82
Sicily	Ounce	123,54	124,80
“	Scudo, or Crown	49,02	49,92
Spain	Real of old plate	4,88	4,57
“	Real of new plate	5,18	4,86
“	Real of Mexican plate	6,48	6,07
“	Real Vellon	2,59	2,43
“	Dollar of old plate, or of exchange	39,	37,30
Stralsund	Rixdollar of account	28,35	variable.
“	Pomeranian Gulden	14,18	“
Strasbourg	Livre and Franc. (See France)		
“	Florin	19,08	18,76
Sweden	Rixdollar	55,41	56,43
Switzerland	Franc (new system)	22,14	——
Trieste	Florin, Austrian currency	25,20	25,05
“	Lira, Trieste currency	4,76	4,73
“	Lira di Piazza	4,65	4,63
Turin	Lira	11,28	11,23
Valencia	Libra	39,45	37,59
Venice	Lira piccola (in the old coins)	5,07	variable.
“	Lira piccola (in Austrian coin)	4,25	“
Vienna	Florin	25,20	25,05
Zante	Lira	4,06	variable.
Zurich	Florin, money of exchange	25,85	“
“	Florin, current	23,50	“

548. The Course of Exchange is the variable price of the money of one country which is given for a fixed amount of that of another country. Thus, the Exchange Par between the United States and England being 54 d. Sterling, when this price is given for the Dollar of the U. S., the Course of Exchange is said to be at Par. When more than 54 d. is given, the Course is *above par*. When less than 54 d. is given, the Course is *below par*.

549. The sum of money for which a Bill of Exchange is drawn, is expressed in the money of the place on which it is drawn. Thus, when a merchant in the United States draws on his debtor or banker in London, his Draft is styled *Bill on London*, or *United States on London*, and the amount of the Bill is expressed in *Sterling Money*.

550. When the Course is above or below par, the variation in the uncertain price is generally rated at so much per cent.

To find the value of a Bill.

This, in the case of a Direct Exchange, and where the Course is at par, is a mere Reduction of Currencies, and needs no farther explanation.

551. When the Course is q per cent. above par, the Exchange is evidently q per cent. in favour of the Place which gives the *Certain Price;* consequently, to reduce the *Certain Money* to the *Uncertain*, the amount which would result at par must be multiplied by $\frac{100 + q}{100}$; and inversely, to reduce the *Uncertain Money* to the *Certain*, the result at par must be multiplied by $\frac{100}{100 + q}$.

552. When the Course is q per cent. below par, the Exchange is q per cent. in favour of the Place which gives the *Uncertain Price;* wherefore, to reduce the *Uncertain Money* to the *Certain*, the *par result* must be multiplied by $\frac{100 + q}{100}$; and inversely, to reduce the Certain Money to the Uncertain, the *par result* must be multiplied by $\frac{100}{100 + q}$.

London on United States.

1. Reduce $5840,50 to Sterling, exchange at 4 p. c. above par.

As the United States gives the *Certain Price*, the London merchant finds the Sterling worth of his Bill in multiplying the dollars by $\frac{100 + q}{100} \times \frac{9}{40}$. Thus,

$5840{,}50 \times \frac{104}{100} \times \frac{9}{40} = 5{,}8405 \times 26 \times 9 = 11{,}681 \times 13 \times 9 = 11{,}681 \times 117 =$ £1366,677 = £1366 13 s. 6½ d.—.

United States on London.

2. Reduce £5513 8 s. to Federal Money, exchange at 4 p. c. above par.

$5513\frac{2}{5} = \frac{27567}{5}$; then $\frac{27567}{5} \times \frac{40}{9} \times \frac{100}{104} = \frac{3063 \times 200}{26} = \frac{306300}{13} =$ \$23561,53$\frac{1}{13}$.

3. Reduce \$3578,20 to Sterling, at 2 per cent. above par.
Ans. £821 3 s. 11¼ d. +.

4. Reduce £1370 6 s. 8 d. to Federal Money, at 4 p. c. below par. Ans. \$$6333{,}9\dot{8}5\dot{1}$, or \$6333,99 —.

5. Reduce £1366 13 s. 6½ d. to Federal Money, at 4 per cent. above par. Ans. \$5840,50 +.

6. Reduce \$6333,99 to Sterling, at 4 p. c. below par.
Ans. £1370 6 s. 8 d. +.

Arbitration of Exchange.

553. Simple Arbitration is the method of finding the Course of Exchange between two places proportioned to prices quoted between each of them and a third place. This course is called *The Arbitrated Price.*

1. Suppose the Course between London and Paris to be 24 Francs for £1 Sterling, and between Paris and Philadelphia 5 Francs, 40 Centimes per Dollar; what is the *Arbitrated Price* between London and Philadelphia?

$$\overset{\text{Fr.}}{5}, \overset{\text{C.}}{40} = 5\tfrac{2}{5} = \overset{\text{Fr.}}{\tfrac{27}{5}}$$

$$\overset{\text{Fr.}}{\tfrac{27}{5}} : \overset{\text{Fr.}}{24} :: \$1, \text{ or multiplying the first ratio by } \tfrac{5}{3},$$

$$9 : 40 :: \$1 : \tfrac{40}{9} = \$4{,}44\tfrac{4}{9}, \text{ Arb. Price per £1 Sterling.}$$

Here we say the value of \$1, in Francs, is to the value of £1 Sterling in Francs, so is \$1 to the number of Dollars equivalent to £1 Sterling.

Also, $\overset{\text{Fr.}}{24} : \overset{\text{Fr.}}{\tfrac{27}{5}} :: \overset{\text{£.}}{1}$; multiplying the first ratio by $\frac{5}{3}$,

$40 : 9 :: \overset{\text{£.}}{1} : \overset{\text{£.}}{\tfrac{9}{40}} = 4$ s. 6 d., Arb. Price per Dollar.

By the Chain Rule.

£1 Sterling $= \overset{\text{£1.}}{24}$ Francs,	1 Dollar $\overset{\text{\$1.}}{=} \frac{27}{5}$ Francs,
$\frac{27}{5}$ Francs $=$ 1 Dollar,	24 Francs $=$ 1 Pound Sterling,
Reduced gives $\frac{40}{9} = \$4{,}44\frac{4}{9}$.	Reduced gives $\$\frac{9}{40} = 4$ s. 6 d., as before.

2. Suppose the Course of Exchange between the United States and Amsterdam to be 42 cents per Florin or Gilder, and between Amsterdam and Paris 55 Grotes Flemish for 3 Francs; what is the Arbitrated Price between the United States and Paris?

100 Cents,
42 Cents = 1 Florin,
6 Florins = 20 Shillings Flemish,
1 s. Flemish = 12 Grotes,
55 Grotes = 3 Francs.

Reduced, gives the Arb. Price per Dollar 5 Francs, $19\frac{1}{2}$ Centimes nearly.

Now suppose that N of the United States draws direct on Paris for 2700 Francs, the actual price being 5 Fr. 40 C., his Bill is worth \$500, which is found thus:

$$\frac{2700}{5{,}4} = \frac{1000}{2} = 500.$$

But if he orders his correspondent in Amsterdam to draw on Paris and remit the value to the U. S., his bill will, by the Indirect or Circular Exchange, be worth $\frac{2700}{5{,}195} = \frac{540000}{1039}$ $= \$519{,}73$.

Allowance, however, must be made for delay and difference of charges between direct and indirect operations.

On the contrary, if N of U. S. remits directly to Paris, he will, for 1 dollar, receive 5 Fr. 40 C.; whereas, by remitting through Amsterdam, he will only recieve 5 Fr. $19\frac{1}{2}$ C.

3. Suppose the Course of Exchange between the United States and Hamburg to be 35 cents per Mark Banco, and between Hamburg and Paris $25\frac{1}{8}$ s. banco for 1 Ecu of 3 Francs; what is the Arb. Price between the United States and Paris?

100 Cents,
35 Cents = 1 Mark bco,
1 Mark = 16 Shillings bco,
$25\frac{1}{8}$ s. bco. = 3 Francs,

which gives the Arb. Price 5 Fr. $45\frac{4}{5}$ C. per Dollar of the U. S.

Suppose N of U. S. draws on Paris for 2700 Francs, through Hamburg, his Bill will be worth $\frac{2700}{5,458}$ = $494,68, which is less than by direct exchange.

Whereas, if he remits to Paris through Hamburg, he will receive 5 Fr. $45\frac{4}{5}$ C. per Dollar, which is more than by direct exchange.

554. From these results, and those of the preceding example, it is plain that the interest of the Drawer is identified with that of *the Place on which he draws*, and consequently, directly opposite to that of the Remitter, which is identified with that of *the Place where he resides.* Hence also is deduced the following

General Rule.

Draw upon the place where the arbitrated price is better for that place than the advised price, and remit to the place where it is worse.

555. In like manner also are deduced the two following

Special Rules.

(1.) *On a place to which the United States gives the Certain Price, the Bill should be drawn at the lowest course and remitted at the highest.*

(2.) *On a place to which the United States gives the Uncertain Price, a Bill should be drawn at the highest course and remitted at the lowest.*

The above Rules, taken as is almost the whole of Exchange, from the Cambist, in which we have substituted the United States for London, will evidently apply to any other places.

4. The course of exchange between the U. S. and Cadiz being 68 cts. per Dollar of Exchange, or of Plate, and between Cadiz and Paris 14 Fr. 45 C. per Doubloon of 4 dollars of Plate; what is the arbitrated price between the U. S. and Paris?

100 Cents,
68 Cents = 1 Dollar of Plate,
4 Dol. Plate = 14 Francs, 45 Centimes.

Arb. Price, 5 Fr. $31\frac{1}{4}$ C.

5. The exchange between the U. S. and Lisbon being $1,24 per Milree, and between Lisbon and Paris 520 Rees per Ecu of 3 Fr.; what is the arbitrated price between U. S. and Paris?

100 Cents,
124 Cents = 1000 Rees,
520 Rees = 3 Francs.

Arb. Price, 4 Fr. 65¼ C.

6. The exchange between the U. S, and Genoa being 89 Cents per Pezza, and between Genoa and Paris 465 Centimes per Pezza, what is the arbitrated price between U. S. and Paris?

100 Cents,
89 Cents = 1 Pezza,
1 Pezza = 465 Centimes.

Arb. Price, 5 Fr. 22½ C.

7. The exchange between the U. S. and Leghorn being 90 Cents per Pezza, and between Leghorn and Paris 503 Centimes per Pezza, what is the arbitrated price between U. S. and Paris?

Arb. Price, 5 Fr. 58⁸⁄₉ C.

8. The exchange between U. S. and Venice being 13½ Cents per Lira Piccola, and between Venice and Paris 105 Centimes of Venice for 1 Franc, what is the arbitrated price between U. S. and Paris?

100 Cents,
13½ Cents = 1 Lira Piccola,
40000 Lire Piccole = 30467 Lire Italiane,
1 Lira Italiana = 100 Centimes of Venice,
105 Centimes = 1 Franc.

Arb. Price, 5 Fr. 37⅓ C.

United States and Paris.

556. Recapitulation of the preceding arbitrated prices, with directions for drawing and remitting to the greatest advantage.

The arbitrated price through Amsterdam 5 Fr. 19½ C.
" through Hamburg, 5 45⅘
" through Cadiz, 5 31¼
" through Lisbon, 4 65¼ lowest.
" through Genoa, 5 22½
" through Leghorn, 5 58⁸⁄₉ highest.
" through Venice, 5 37⅓
And the direct course of U. S. on Paris is 5 40

Now by Rule 1. (555) As the United States gives the *Certain Price* to Paris, that is, one Dollar for a variable number of Francs; the most advantageous place to draw through, is Lisbon, which gives the lowest price, and to remit through is Leghorn, which gives the highest.

Also, if N of U. S. has to draw on Paris, any one of the other places, except Hamburg, is more advantageous than the direct course, and to remit through Hamburg is more advantageous than the direct course.

557. The gain or loss per cent. resulting from one mode of operation, instead of another, as, for example, drawing through Lisbon, instead of directly on Paris, may be thus determined:

$$465\tfrac{1}{4} : 540 : : 100 : 116\tfrac{1}{15} \text{ nearly.}$$

Hence the gain is nearly $16\frac{1}{15}$ per cent.

Suppose N of the U. S. sells a Draft on Lisbon of 403 Milreas $225\frac{4}{5}$ Rees, for which, at $1,24 per Milree, he receives $500, and that he buys a Bill on Paris of 2326 Francs $30\frac{1}{4}$ Centimes, which at 520 Rees per Ecu of 3 Francs, is the equivalent of his Draft on Lisbon, and for which, at 5 Fr. 40 C. per Dollar, he pays $430,80; it is plain that the money he receives for his Draft will pay for the Bill he remits with a surplus of $69,20, and that his debit in Lisbon is balanced by his credit in Paris.

Proof. 100 : 124 : : 540 : 6 Francs $69\frac{3}{5}$ Centimes, the arbitrated price per Milree. Then,

$403{,}2258 \times 6{,}696 = 2699{,}9999568$, or 2700 Francs; and $2700 \div 5{,}40 = \$500$.

Also, 100 : $116\frac{1}{15}$: : $430,80; 15 : 17,41 : : $430,80. Or finally 1 : 3,482 : : 143,6 : $500,0152, or $500 very nearly.

558. The operation of drawing on one place through another, as on Amsterdam through Paris, may be performed in three different ways:

1. The United States may draw on Paris, and order his correspondent there to draw on Amsterdam.
2. The United States may draw on Paris, and order his correspondent in Amsterdam to remit the same sum to Paris.
3. The U. S. may order his correspondent at Paris to draw on Amsterdam, and remit the value to the United States.

559. The operation of remitting to one place through another, as to Amsterdam through Paris, may also be performed in three different ways:

1. The U. S. may remit to Paris, and order his correspondent there to remit the sum to Amsterdam.

2. The U. S. may remit to Paris, and order his correspondent in Amsterdam to draw on Paris.

3. The U. S. may take Bills on Paris, and remit them to Amsterdam, there to be negotiated.

Use of Logarithms and Fixed Numbers in Bullion and Exchange Operations.

560. In the reduction of a statement of the Chain Rule, it is evident that, if, from the log. of the sum of the Consequents, we subtract the log. of the sum of the Antecedents, the difference will be the log. of the answer.

Or, to the log. of the sum of the Consequents add the arithmetical complement of the log. of the sum of the Antecedents, rejecting ten in the index.

561. Fixed Numbers are of great utility in exchange, whether used in Logarithms or not.

In long statements by the Chain Rule, there are several constant or invariable proportions, as the number of shillings in a Pound, the established Mint Prices of Gold and Silver, &c. These may, by reduction, be converted into one number, which is called *Fixed.*

Gold in Bars.

To calculate the Course of Exchange from the Price of Bullion, and the contrary. Also, to find Fixed Numbers for the solution of such questions.

1. When gold is sold at Hamburg at 104 Sols banco per Ducat of $23\frac{1}{2}$ Carats fine, and at London 78 s. 2 d. per standard ounce; what should be the Course of Exchange, supposing that 67 Ducats make 1 Cologne Mark fine, and that 60 Marks weigh 451 Ounces Troy?

1 Pound Sterling,
1 Pound Sterling = 20 Shillings,
$78\frac{1}{6}$ Shillings Sterling = 1 Ounce Standard,
451 Ounces Standard = 60 Marks Standard,
47 Marks $\frac{44}{48}$ = 44 Marks $\frac{47}{48}$,
1 Mark = 67 Ducats,
1 Ducat = 104 Shillings Lubs Banco;
6 Shillings Lubs Bo. = 1 Shilling Flemish.

As the prices of gold are the only variable numbers in the above statement, the Fixed Number may be thus determined:

$$\frac{20 \times 60 \times 44 \times 67}{451 \times 47 \times 6} = 27{,}815, \text{ the Fixed Number.}$$

Hence, $27{,}815 \times 104 \div 78\frac{1}{8} = 37$ Shillings, Flemish, (the Course of Exchange,) for £1 Sterling.

By the above Fixed Number, with the Course of Exchange, and the price of bullion in one place, it is plain that its price may be found in the other place.

2. When gold is sold in Amsterdam at an advance of 17 per cent. on the Tariff price of 355 Florins per Mark fine, and in London at 78 s. per standard Ounce, what is the Course of Exchange, supposing that 80 Marks of Amsterdam weigh 633 Ounces English Troy?

1 Pound Sterling,
1 Pound Sterling = 20 Shillings,
78 Shillings = 1 Ounce Standard,
12 Ounce Standard = 11 Ounces fine,
633 Ounces fine = 80 Marks of Amsterdam,
1 Mark = 355 Florins,
100 Florins = 117 with the premium.

The invariable numbers in the above statement are

$$\frac{20 \times 11 \times 80 \times 355}{12 \times 633 \times 100} = 8{,}225, \text{ the Fixed Number, which}$$

$\times 117 \div 78 = 12$ Fl. $6\frac{3}{4}$ St., or 41 s. 1 d. Flemish per Pound Sterling, the Course of Exchange.

3. What is the Course of Exchange between Paris and London resulting from the price of gold: viz. 8 per Mille premium on the Tariff price, which is 3434 Fr. 44 C. per Kilogramme fine, and 78 s. per Ounce English standard?

1 Pound Sterling,
1 Pound Sterling = 20 Shillings,
78 Shillings = 1 Ounce Standard,
12 Ounces Stand. = 11 Ounces fine,
32,154 Ounces fine = 1 Kilogramme fine,
1 Kilogramme fine = 3434 F. 44 C. Tariff price,
100 Francs = 100,8 with the premium.

The invariable numbers in the above statement, are

$$\frac{20 \times 11 \times 3434\,f.\,44}{12 \times 32{,}154 \times 100} = 19{,}5822, \text{ the Fixed Number.}$$

Hence, $19{,}582 \times 100{,}8 \div 78 = 25$ Fr. 30 C. per Pound Sterling, the Course of Exchange.

4. If the price of gold at Cadiz be 30 Reals per Castellano

of $22\frac{1}{2}$ Carats fine, and in London $934\frac{1}{2}$ d. per ounce standard, (the Mint price,) what is the Course of Exchange, &c.?

1 Piastre,
1 Piastre = 8 Reals,
30 Reals = 1 Castellano,
44 Castellanos $\frac{45}{48}$ = 45 Castellanos $\frac{44}{48}$,
250 Castellanos = 37 Ounces Troy,
1 Ounce = $934\frac{1}{2}$ Pence.

Reduced, gives the Course of Exchange $37\frac{3}{4}$ d. per Piastre.

The invariable Numbers are $\frac{8 \times 45 \times 37}{44 \times 250} = 1{,}211$, the Fixed Number.

5. If gold be sold at Lisbon for 1700 Rees per Outava of $\frac{22}{24}$ fine, and in London at the Mint price, what is the Course of Exchange?

1 Milree,
1 Milree = 1000 Rees,
1700 Rees = 1 Outava,
64 Outavas = 1 Mark,
8 Marks = 59 Ounces Troy,
1 Ounce = $934\frac{1}{2}$ Pence.

Reduced, gives $63\frac{1}{3}$ d. per Milree, the Course of Exchange.

The invariable Numbers are $\frac{1000 \times 59}{64 \times 8} = 115{,}234$, the Fixed Number.

6. If the price of gold at Leghorn be 108 Lire per Ounce fine, what is the Course of Exchange, &c.?

1 Pezza,
4×1 Pezza = $5\frac{3}{4}$ Lire $\times 4 = 23$,
108 Lire = 1 Ounce fine of Leghorn,
144 Oz. of Leghorn = 131 Ounces of London,
11 Oz. fine = 12 Ounces Standard,
1 Oz. Standard = $934\frac{1}{2}$ Pence.

Reduced, gives 49,37 d. per Pezza, the Course of Exchange.

The invariable numbers are $\frac{23 \times 131 \times 12}{4 \times 144 \times 11} \doteq 5{,}706$, the Fixed Number.

The foregoing Examples solved by Logarithms.

Hamburg,	Fixed Number....	27,815	Log.	1,44428
	Hamburg Price..	104	Log.	2,01703
	London Price.....	$78\frac{1}{6}$	Ar. Comp...	8,10698
	Course of Exchange	37,08	Log.	$\cancel{1}$ 1,56829

Amsterdam,	Fixed Number	8,225	Log.	0,91514
	Amsterdam Price	117	Log.	2,06819
	London Price	78	Ar. Comp.	8,10791
	Course of Exchange	12,338	Log.	1,09124
Paris,	Fixed Number	19,582	Log.	1,29186
	Paris Price	100,8	Log.	2,00346
	London Price	78	Ar. Comp.	8,10790
	Course of Exchange	25,30	Log.	1,40322
Cadiz,	Fixed Number	1,201	Log.	0,08310
	London Price	934,5	Log.	2,97058
	Cadiz Price	30	Ar. Comp.	8,52288
	Course of Exchange	37,72		1,57656
Lisbon,	Fixed Number	115,23	Log.	2,06156
	London Price	934,5	Log.	2,97058
	Lisbon Price	1700	Ar. Comp.	6,76955
	Course of Exchange	63,34	Log.	1,80169
Leghorn,	Fixed Number	5,706	Log.	0,75633
	London Price	934,5	Log.	2,97058
	Leghorn Price	108	Ar. Comp.	7,96658
	Course of Exchange	49,37	Log	1,69349

In Bullion operations, the following proportions will be found useful:

Eng. Oz. Troy.
60 Marks of Hamburg or Cologne = 451
80 Marks of Amsterdam = 633
61 Oz. Paris, Poids de Marc = 60
1 Kilogramme = 32,154
31,1 Grammes = 1
5 Marks of Spain, or 250 Castellanos = 37
8 Marks of Portugal = 59
144 Oz. of Leghorn or Florence = 131
100 Oz. of Naples = 86
11 Oz. of Rome = 10

Oz. Eng. Stand. fineness.
1000 Span. Dollars = 866 Oz. Troy = 835
1000 Doubloons = 868 Oz. Troy = 853
1000 Joannese, or Ports. = 460
55 English ounces French Gold = 54
37 Ditto French Silver = 36
19 Marks fine Silver, Amsterdam = 164

	Oz. Eng. Stand. fineness.
34 Marks of Hamburg Ducats	= 273
8 Marks fine Silver of Hamburg	= 65
11 Oz. of Dollars	= 107
48 Oz. of do.	= 43 oz. of fine Silver.

Silver in Bars.

562. To find the Course of Exchange between London and the foregoing places, resulting from the following prices of silver:

Hamburg.......	27 Marks Banco, per Cologne Mark fine.
Amsterdam....	26 Florins, per Mark fine.
Paris............	3 p. c. Premium on the Tariff price of 218 f. 89 c. per Kilogramme fine.
Cadiz............	108 Reals of Plate, per Mark fine.
Lisbon..........	990 Rees, per Ounce fine.
Leghorn........	84 Lire, per Libra of 12 Ounces fine.
London..........	58 Pence, per Ounce $\frac{37}{40}$ fine.

Statements and Results.

		Course of Ex.	Fix'd Num.
Hamburg,	$\overset{\text{£ Ster.}}{\frac{1}{1}} \times \overset{\text{Pence.}}{\frac{240}{58}} \times \overset{\text{Oz. Stan.}}{\frac{1}{40}} \times \overset{\text{Oz. fine.}}{\frac{37}{451}} \times \overset{\text{Mks.}}{\frac{60}{1}} \times \overset{\text{Mks. Bo.}}{\frac{27}{3}} \times \overset{\text{s. Fl.}}{\frac{8}{\ }} =$	36,66	78,758.
Amsterdam,	$\frac{1}{1} \times \frac{240}{58} \times \frac{1}{40} \times \frac{37}{638} \times \frac{80}{1} \times \frac{\text{26 Flor.}}{\ } =$	12,57	28,057.
Paris,	$\frac{1}{1} \times \frac{240}{58} \times \frac{1}{40} \times \frac{37}{32{,}154} \times \frac{218{,}89}{100} \times \frac{103}{\ } =$	26,83	15,113.
Cadiz,	$\overset{\text{Piastre.}}{\frac{1}{1}} \times \overset{\text{Reals.}}{\frac{8}{108}} \times \overset{\text{Mks.}}{\frac{1}{5}} \times \overset{\text{Mks. fine.}}{\frac{37}{87}} \times \overset{\text{Oz. st.}}{\frac{40}{1}} \times \overset{\text{Pence.}}{\frac{58}{\ }} =$	34,87	64,000.
Lisbon,	$\overset{\text{Pezza.}}{\frac{1}{1}} \times \overset{\text{Lire.}}{\frac{5\frac{3}{4}}{84}} \times \overset{\text{Oz. Leg.}}{\frac{12}{144}} \times \overset{\text{Oz. Tr.}}{\frac{131}{87}} \times \overset{\text{Oz. St.}}{\frac{40}{1}} \times \overset{\text{Pence.}}{\frac{58}{\ }} =$	46,85	67,860.

563. There are three variable numbers in each of the foregoing statements; viz.: the price of Foreign Bullion, the London Price, and the Course of Exchange. Now, it is obvious that if any two of these prices be given, the third may thence be found by the help of the Fixed Number. Suppose, for example, in the last statement respecting Paris, that $F =$ the Fixed Number, $p =$ the Paris price, $l =$ the London price, and $c =$ the Course of Exchange, then c, l, or p, may be found, the rest being known. Thus

$$c = \frac{Fp}{l} = \frac{15{,}113 \times 103}{58} = 26{,}83 \text{ Francs per £1 Sterling.}$$

$$l = \frac{\mathrm{F}p}{c} = \frac{15{,}113 \times 103}{26{,}83} = 58 \text{ Pence per Ounce Sterling.}$$

$$p = \frac{cl}{\mathrm{F}} = \frac{26{,}83 \times 56}{15{,}113} = 103 \text{ Francs per Kilogramme fine.}$$

564. The above Equations will answer for all other places to which London gives the certain, but where it gives the uncertain, as to Cadiz, the following will be the Formulæ, supposing s to represent the Spanish or Cadiz price.

$$c = \frac{\mathrm{F}l}{s} = \frac{64 \times 58}{108} = 34{,}37 \text{ Pence per Piastre.}$$

$$s = \frac{\mathrm{F}l}{c} = \frac{64 \times 58}{34{,}37} = 108 \text{ Reals per Mark.}$$

$$l = \frac{\mathrm{S}c}{\mathrm{F}} = \frac{108 \times 34{,}37}{64} = 58 \text{ Pence per Ounce.}$$

These Formulæ will apply to operations in gold as well as silver; and where the calculations are laborious, Logarithms may be introduced with advantage; but in ordinary cases, common numbers seem more convenient and intelligible.

565. If we ask why the Price, when *above par*, is said to be favourable to the place which gives the certain for the uncertain, and the contrary when *below par*, seeing that the Drawers and Remitters, whose interests are opposed, are natives of the same country, the answer is, that when the Exchange is unfavourable to a place, it is the interest of the Remitters to pay their foreign debts in specie or Bullion, instead of Bills, and the exportation of these is often considered a national disadvantage.

Compound Arbitration

566. Is a comparison between the exchanges of more than three places, in order to find how much a remittance, passing through them all, will amount to in the last place, or to find the arbitrated price between the first place and the last, in order to determine on the most advantageous mode of negotiating bills.

Compound Arbitration may, like Simple Arbitration, of which it is merely an extension, be solved by several proportions, or by the Chain Rule, as in the following example:

1. Suppose the exchange between the U. S. and Amsterdam to be 40 cents per Florin, and between Amsterdam and Hamburg 32⅝ Stivers for 1 Rixdollar of 2 Marks Hamburg

Banco, and between Hamburg and Paris $25\frac{7}{8}$ Schillings Banco for 1 Ecu of 3 Francs; what is the arbitrated price between the U. S. and Paris?

1 Florin : $32\frac{5}{8}$ Stivers :: 40 cents : $65\frac{1}{4}$ cts. = 2 Marks Banco; 2 Marks Bo. : $25\frac{7}{8}$ Sch. Bo. :: $65\frac{1}{4}$ cts. : $52\frac{19}{25}$ cts. = $52\frac{19}{25}$ Sch. Banco; $52\frac{19}{25}$ cents : 100 cents :: 3 Francs : 5 Fr. $68\frac{5}{8}$ cts. Arbitrated Price.

By the Chain Rule.

100 Cents,
40 Cents = 1 Florin,
1 Florin = 20 Stivers,
$32\frac{5}{8}$ Stivers = 2 Marks Banco,
1 Mark Bo. = 16 Schillings,
$25\frac{7}{8}$ Schillings = 3 Francs,

which gives the arbitrated price 5 Francs $68\frac{5}{8}$ Centimes, nearly.

Here, as the U. S. gives the *certain price*, allowing the direct exchange on Paris to be, as in the preceding examples, 5 F. 40 c., it would be more advantageous to draw directly on Paris than by the circular exchange.

2. Suppose London has a sum of money to receive in Cadiz, the exchange being at 35 Shillings Flemish per Pound Sterling; between Amsterdam and Paris, $53\frac{1}{2}$ Grotes Flemish per Ecu of 3 Francs; and between Paris and Cadiz, 15 Francs 50 Centimes per Doubloon of Plate; what is the arbitrated price between London and Cadiz?

1 Dollar of Plate,
4 Dollars of Plate = 1 Doubloon do.,
1 Doubloon = $15\frac{1}{2}$ Francs,
3 Francs = $53\frac{1}{2}$ Grotes Flemish,
12 Grotes = 1 Shilling Flemish.
35 s. Flemish = 240 Pence Sterling.

Reduced, gives $39\frac{1}{2}$ d. Sterling per Dollar of Plate.

The circular operation would, therefore, be most advantageous, as London would receive $39\frac{1}{2}$ d., instead of 38 d., for each Dollar, due to him in Cadiz.

3. London having a sum to receive in Lisbon, when the exchange is at 64 d. Sterling per Milree, draws on Lisbon, but remits his bill to Hamburg to be negotiated, and directs the returns to be made in bills on Leghorn—the exchange between Hamburg and Lisbon being 45 Grotes Flemish per Old Crusade; between Hamburg and Leghorn, 85 Grotes

Flemish per Pezza, and between London and Leghorn 52 d. Sterling per Pezza; what is the arbitrated price between London and Lisbon?

1000 Rees,
400 Rees = 1 Old Crusade,
1 Old Crusade = 45 Grotes Flemish,
85 Grotes = 1 Pezza,
1 Pezza = 52 Pence Sterling.
Reduced, gives $68\frac{14}{17}$ Sterling per Milree.

This circular operation would, therefore, be more advantageous than the direct exchange of London on Lisbon, viz.: $68\frac{14}{17}$ d., instead of 64 d.

4. London has a sum to pay in Petersburg, and another to receive in Genoa; but their being no regular exchange between these places, London draws on Hamburg, and remits his bills to Petersburg, directing Hamburg to draw on Genoa; the exchange between London and Genoa being $46\frac{1}{2}$ d. Sterling per Pezza; between Hamburg and Genoa, 81 Grotes Flemish per Pezza; and between Petersburg and Hamburg, 23 Schillings Lubs per Ruble; what is the exchange between London and Petersburg resulting from the operation; that is, how many Pence Sterling does London pay for the Ruble?

1 Ruble,
1 Ruble = 23 Schillings Lubs,
1 Schillings Lubs = 2 Grotes Flemish,
81 Grotes = 1 Pezza,
1 Pezza = $46\frac{1}{2}$ Pence Sterling,
Reduced, gives $26\frac{11}{27}$d. Sterling per Ruble.

5. London has a sum to remit to Paris, the exchange being at 24 Livres 5 Sous per Pound Sterling; but instead of taking a bill on Paris, London draws on Hamburg, and remits his bill to Paris to be negotiated; Hamburg is directed to draw on Venice, and Venice to draw on London; the exchange between Paris and Hamburg being 190 Francs per 100 Marks Banco, between Hamburg and Venice 4 Lire 5 Soldi Piccoli per Mark Banco, and between Venice and London 55 Lire Piccole per Pound Sterling; what is the arbitrated price between London and Paris?

1 Pound Sterling,
1 Pound Sterling = 55 Lire Piccole,
$4\frac{1}{2}$ Lire Piccole = 1 Mark,

100 Marks = 190 Francs,
80 Francs = 81 Livres.

Reduced, gives 24 Livres 18 Sous per Pound Sterling.

The circular operation is therefore more advantageous than the direct exchange, as London pays, with £1 Sterling, 24 Livres 18 Sous, instead of 24 Livres 5 Sous.

567. Hitherto we have only examined one combination of the exchanges between several places in order to discover one result. In the following example different combinations of the same exchanges are examined, to find which is most favourable; and what is here performed with four places only, may be done with any greater number in the same manner; that is, by trial and comparison.

Suppose the following to be the quotations of exchange:

London on	Amsterdam	35	s. Flemish per Pound Sterling.
"	Madrid.....	38	d. Sterling per Dollar of Plate.
"	Paris........	24	Livres per Pound Sterling.
Amsterdam on	London	34	s. Flemish per Pound Sterling.
"	Paris	53	Grotes per Ecu of 3 Francs.
"	Madrid.....	92	Grotes per Ducat of Plate.
Paris on	London.....	23½	Livres per Pound Sterling.
"	Madrid.....	16	Francs per Doubloon of Plate.
"	Amsterdam.	54	Grotes Fl. per Ecu of 3 Francs.
Madrid on	London....	39	d. Sterling per Dollar of Plate.
"	Amsterdam.	94	Grotes per Ducat of Plate.
"	Paris........	16½	Francs per Doubloon of Plate.

Now if London has a sum to receive in Madrid, what mode of operation will be most advantageous?

By drawing directly on Madrid, London will receive 38 d. Sterling per Dollar.

1. Let London draw on Amsterdam, directing Amsterdam to draw on Paris, and Madrid to remit to Paris; then

1 Dollar of Plate,
4 Dollars of Plate = 1 Doubloon of Plate,
1 Doubloon = 16½ Francs,
3 Francs = 54 Grotes Flemish,
12 Grotes = 1 s. Flemish,
35 s. Flemish = 240 d. Sterling.

Reduced, gives the Dollar equal to $42\frac{3}{7}$ d. Sterling.

2. Let London draw on Paris, directing Paris to draw on Amsterdam, and Madrid to remit to Amsterdam; then

1 Dollar of Plate,
1 Dollar of Plate = 272 Maravedis,
375 Maravedis = 1 Ducat of Plate,
1 Ducat = 94 Grotes Flemish,
53 Grotes = 3 Francs,
80 Francs = 81 Livres,
24 Livres = 240 d. Sterling.

Reduced, gives 39$\frac{1}{13}$ d. per Dollar.

3. Let London draw on Madrid, and remit the bill to Paris to be negotiated, and let the returns be made in a bill on Amsterdam; then

1 Dollar of Plate,
4 Dollars = 1 Doubloon of Plate,
1 Doubloon = 16 Francs,
3 Francs = 54 Grotes Flemish,
12 Grotes = 1 s. Flemish,
35 s. Flemish = 240 d. Sterling.

Reduced, gives 41$\frac{1}{7}$ d. Sterling per Dollar.

4. Let London draw on Madrid, and remit to be negotiated in Amsterdam, and order the returns to be made in a bill on Paris; then

1 Dollar of Plate,
1 Dollar = 272 Maravedis,
375 Maravedis = 1 Ducat of Plate,
1 Ducat = 92 Grotes Flemish,
53 Grotes = 3 Francs,
80 Francs = 81 Livres,
24 Livres = 240 d. Sterling.

Reduced, gives 38$\frac{1}{4}$ d. Sterling per Dollar, nearly.

5. Let London draw on Amsterdam, Amsterdam on Paris, and Paris on Madrid; then

1 Dollar of Plate,
4 Dollars = 1 Doubloon of Plate,
1 Doubloon = 16 Francs,
3 Francs = 53 Grotes Flemish,
12 Grotes = 1 s. Flemish,
35 s. Flemish = 240 d. Sterling.

Reduced, gives 40$\frac{8}{21}$ d. per Dollar.

6. Let Madrid remit to Paris, Paris to Amsterdam, and Amsterdam to London; then

1 Dollar of Plate,
4 Dollars = 1 Doubloon of Plate,
1 Doubloon = $16\frac{1}{2}$ Francs,
3 Francs = 54 Grotes Flemish,
12 Grotes = 1 s. Flemish,
34 s. Flemish = 240 d. Sterling.
Reduced, gives $43\frac{3}{4}$ d. per Dollar, nearly.

Several other combinations might be made, but the foregoing are sufficient for the present illustration.

In recapitulating the different combinations here proposed, we find that

The First	gives	$42\frac{3}{7}$	Pence Sterling	per Dollar of Plate.
The Second	"	$39\frac{1}{13}$	"	per Ditto.
The Third	"	$41\frac{1}{7}$	"	per Ditto.
The Fourth	"	$38\frac{1}{4}$	"	per Ditto.
The Fifth	"	$40\frac{8}{21}$	"	per Ditto.
The Sixth	"	$43\frac{3}{4}$	"	per Ditto.
While the direct course	"	38	"	per Ditto.

Hence, it is evident, according to the rules laid down, (555,) that the 6th operation would be the most advantageous for the London drawer, as he would receive more Pence for the Dollar than by any other combination; but it must be observed that in this case, he would have to wait about six months for the remittance from Madrid through Paris and Amsterdam; whereas, by drawing directly on Madrid, he would receive the money immediately; consequently, interest should be deducted from the profit, besides commission, postage, &c. The Profit is, however, so great in this case, as far to exceed the additional expenses; thus:

$$38 : 43\tfrac{3}{4} :: 100 : 115\tfrac{5}{38}. \quad \text{Profit. } 15\tfrac{5}{38} \text{ per cent.}$$

It is also evident, that if London was the debtor, and had to remit to Madrid, the lowest course of exchange, which is here the direct one, should be adopted.

Compound Interest.

568. When the interest of a sum of money, at the time it is due, is, instead of being paid, added to the principal, and the sum considered a new principal on which interest is computed as before, and again when due, added to form a new

principal, and so on, the accumulation of interest thence arising, is called Compound Interest.

This accumulation is so much the greater as the increment of time for which it is calculated is shorter.

Let P be the principal and r the rate p. c. per annum; then $100 : 100 + r :: P : \frac{100 + r}{100} \times P$, the amount for one year. Hence we say: Multiply the principal by the fraction $\frac{100 + \text{the rate}}{100}$, reduced to its lowest terms, or to a decimal, or otherwise, as may be most convenient.

The product or amount, which is the new principal, multiplied by $\frac{100 + r}{100}$, will evidently give the amount for the second year; and continuing thus, it is plain that $P \times \frac{100 + r}{100}$ = amount for 1 year; $P \times \left(\frac{100 + r}{100}\right)^2$ = amount for 2 years; $P \times \left(\frac{100 + r}{100}\right)^3$ = amount for 3 years, &c. Hence the General Rule: Multiply the principal by $\frac{100 + r}{100}$ raised to a power signified by the number of years, half years, or other portions of time, for which the interest is calculated.

1. What is the compound interest of \$340 for 5 years at 5 p. c.?

$\frac{105}{100} = \frac{21}{20}$, and $\left(\frac{21}{20}\right)^5 = \frac{4084101}{3200000}$; then $\frac{4084101 \times 340}{3200000}$ $= \frac{408,4101 \times 17}{16} = \$433,94$ —.

Lastly, 433,94 — 340 = \$93,94, the interest required.

Proof. 5 p. c. $= \frac{5}{100} = \frac{1}{20}$; then

$340 + \frac{340}{20} = 340 + 17 = \$357$ Amount for 1 year.
$357 + \frac{357}{20} = 357 + 17,85 = 374,85$ " 2 years.
$374,85 + \frac{37,485}{2} = \$393,592$ " 3 years.
$393,592 + \frac{39,3592}{2} = \$413,272$ " 4 years.
$413,272 + \frac{41,3272}{2} = \$433,94$ " 5 years.

Or, *Multiply successively by the decimal amount of* 1 *unit.*

Thus: 340 × 1,05 = \$357 Amount for one year; then

$$\begin{array}{r l l}
3\,57 & & \\
\underline{1,05} & & \\
374,85 & \text{Amount for} & \text{2 years.} \\
\underline{1,05} & & \\
393,59\,25 & \text{“} & \text{3 years.} \\
\underline{1,05} & & \\
413,2721\,25 & \text{“} & \text{4 years.} \\
\underline{1,05} & & \\
\$433,935731\,25 & \text{“} & \text{5 years.}
\end{array}$$

Or, multiply the logarithm of the amount of 1 unit by the number of years or times specified, and to the product add the log. of the principal, which will give the log. of the amount. Thus:

$$\begin{array}{l r l}
l.\ 1,05 \ldots\ldots\ldots 0,021189 \times 5 & = & 0,105945 \\
l.\ 340 & = & \underline{2,531479} \\
\text{log. of the amount} & & 2,637424
\end{array}$$

This log. gives the amount $433,934, which is very nearly the true result.

The student will see, that to multiply the log. of 1,05 by 5, is (474) the same as to raise $\frac{21}{20}$ to the 5th power. Also (472), that to add the log. of 340, is the same as to multiply $(\frac{21}{20})^5$ by that number.

2. What is the compound interest of $340 for 5 years at 5 p. c., the interest payable half-yearly. Ans. $95,23.

The student may verify his results as in Ex. 1.

3. What is the amount of £587 16 s. Sterling at compound interest for 6 yrs. 9 mos. at 6 p. c.?

Ans. £871 6 s. $6\frac{1}{2}$ d. —.

*Use of Geometrical Progression in the Calculation of Compound Interest.**

569. From the preceding article, where P is assumed as the principal, and r the rate p. c., it is evident that $\div\!\!\div\ P : P \times \left(\frac{100+r}{100}\right) : P \times \left(\frac{100+r}{100}\right)^2 : P \times \left(\frac{100+r}{100}\right)^3$, &c., is a geometrical progression. Hence we infer, that taking the principal for the first term, every series in the calculation of compound interest is a geometrical progression.

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*See Table, art. 465.

Let the capital be \$1200, and the rate 10 p. c., or $\frac{1}{10}$ per Dollar. Then, as $\frac{110}{100}=\frac{11}{10}$, the series becomes $\div\!\div\ 1200 : 1200\times\frac{11}{10} : 1200\times(\frac{11}{10})^2 : 1200\times(\frac{11}{10})^3$, &c., of which the first term $a=1200$, the quotient $q=\frac{11}{10}$, and any term ω equals $1200\times(\frac{11}{10})^{n-1}$.

Therefore, calling a the first term, and r the interest of a dollar, the quotient will be $1+r$, and we shall have $\div\!\div\ a : a\times(1+r) : a\times(1+r)^2 : a\times(1+r)^3 \ldots\ldots \omega = 1200\times(1+r)^{n-1}$.

Increase of Population.

570. What is here said of compound interest, applies, word for word, to the increase of population.

To make use of the Table (465) for resolving all problems relative to Compound Interest and the Increase of Population, we shall substitute $1+r$ for q. Thus, instead of the formulæ $q=\sqrt[n-1]{\frac{\omega}{a}}$, $n=1+\frac{L\omega-La}{Lq}$ we shall, for the first, have $1+r=\sqrt[n-1]{\frac{\omega}{a}}$; that is to say, $r=-1+\sqrt[n-1]{\frac{\omega}{a}}=-1+\frac{L\omega-La}{L(n-1)}$, and for the last, $n=1+\frac{L\omega-La}{L(1+r)}$.

Problem 1.

We have placed at interest \$1000000, at the rate of ten per cent. compound interest. What sum shall we touch at the end of five years and a half?

The sum that we shall touch at the end of the fifth year, will equal aq^5. The interest for the six remaining months will equal the half of the interest of aq^5 during a year; that is to say, $aq^5\times\frac{1}{10}\times\frac{1}{2}$; therefore, the sum that we shall touch at the end of five years and a half will equal $aq^5+aq^5\times\frac{1}{10}\times\frac{1}{2}=1000000(\frac{11}{10})^5+1000000(\frac{11}{10})^5\times\frac{1}{10}\times\frac{1}{2}=\$1691035\frac{1}{2}$.

Problem 2.

We have touched \$1610510 for a capital, placed five years since, the compound interest being one-tenth per dollar; what was this capital?

Since we know the last term $1610510(\omega,)$ the number of terms $6(n)$, the quotient $\frac{11}{10}(q)$, and that $a=\frac{\omega}{q^{n-1}}$, we shall have $a = \dfrac{1610510}{(\frac{11}{10})^{6-1}} = \dfrac{1610510}{\frac{11^5}{10^5}} = \dfrac{1610510 \times 10^5}{11^5}$ $= \dfrac{1610510 \times 100000}{161051} = 1000000$ dollars, the required capital.

Problem 3.

We have touched \$1691035½ for a capital, at interest for five years and a half, the compound interest being at ten per cent. What was this capital?

Let a be the capital, the sum touched at the end of the fifth year will be $a \times (\frac{11}{10})^5$, the interest for the six remaining months will equal the half of the interest of $a(\frac{11}{10})^5$ during a year; that is to say, $a(\frac{11}{10})^5 \times \frac{1}{10} \times \frac{1}{2}$, or $a(\frac{11}{10})^5 \times \frac{1}{20}$. We shall then have $a(\frac{11}{10})^5 + a(\frac{11}{10})^5 \times \frac{1}{20} = \omega$; that is to say, $a \times [\,(\frac{11}{10})^5 + (\frac{11}{10})^5 \times \frac{1}{20}\,] = \omega$; therefore,

$$a = \frac{\omega}{(\frac{11}{10})^5 + (\frac{11}{10})^5 \times \frac{1}{20}} = \frac{1691035 + \frac{1}{2}}{(\frac{11}{10})^5 + (\frac{11}{10})^5 \times \frac{1}{20}} = \frac{1691035 + \frac{1}{2}}{\frac{161051}{100000} + \frac{161051}{100000} \times \frac{1}{20}} = \frac{1691035\frac{1}{2}}{\frac{3221020 + 161051}{2000000}} = \frac{(1691035 + \frac{1}{2})2000000}{3382071} = \frac{3382071 \times 1000000}{3382071} = 1000000$$

dollars, the required capital.

Problem 4.

We have put to interest \$1000000, and have touched \$1610510 at the end of five years. What was the compound interest?

Since we know the first term 1000000(a), the last term 1610510(ω), the number of terms 6(n), and that $q = \sqrt[n-1]{\frac{\omega}{a}}$, we shall have $q = \sqrt[5]{\frac{1610510}{1000000}} = \frac{11}{10}$; therefore, the interest was one-tenth per dollar, or ten per 100.

Problem 5.

Having placed at interest \$1000000 at 10 per cent., we have touched \$1691035½. We demand what time the sum was at interest?

Since we know the first term 1000000(a), the last term

$1691035\frac{1}{2}(\omega)$, and the rate, which is one-tenth per dollar, the quotient q will then be $\frac{11}{10}$. But the number of terms $n = 1 + \frac{L\omega - La}{Lq}$; therefore $n = 1 + \frac{L(1691035 + \frac{1}{2} - L1000000}{L11 - L10}$ $= 6\frac{21188}{41393}$.

But the sum touched at the end of the fifth year is equal to $1000000(\frac{11}{10})^5$, that is to say, 1610510; therefore, $1691035 + \frac{1}{2} - 1610510$; that is to say, $80525\frac{1}{2}$ is the interest of 1610510 during a time p of the sixth year. But this interest is equal to $1610510 \times \frac{1}{10} \times p$; that is to say, $161051 \times p$; therefore, $161051 \times p = 80525 + \frac{1}{2}$; therefore, $p = \frac{80525 + \frac{1}{2}}{161051} = \frac{1}{2}$. The sum was therefore at interest $5\frac{1}{2}$ years.

Problem 6.

Suppose that there are 30000000 of inhabitants in France, and that the population augments by one hundredth per year: how many inhabitants will there be in France at the end of a hundred and one years?

Agreeably to the formula $\omega = aq^{n-1}$; we shall have

$$\omega = 30000000 \times (\tfrac{101}{100})^{100}.$$

Operating by logarithms, we shall find for the logarithm of $(\frac{101}{100},)$ $2{,}00432 - 2{,}00000$; that is to say, 0,00432; and for the logarithm of $(\frac{101}{100},)^{100}$, 0,43200, which corresponds to the number $\frac{2704}{1000}$. Therefore $\omega = 30000000 \times \frac{2704}{1000} = 30000 \times 2704 = 81120000$.

Therefore, at the end of a hundred and one years, France will contain 81120000 inhabitants.

Problem 7.

Suppose that in France there are 30000000 of inhabitants, and that the population augments by one hundreth per year: in how many years will the population amount to 81120000 inhabitants?

According to the formula $n = 1 + \frac{L\omega - La}{L(1+r)}$,

we shall have $n = 1 + \frac{L81120000 - L30000000}{L101 - L100}$;

then, as $L81120000 - L30000000 = 0{,}43200$, and $L101 - L100 = 0{,}00432$, we shall have $n = 1 + \frac{0{,}43200}{0{,}00432} = 1 + 100 = 101$.

Therefore, in one hundred and one years the population of *France* will be 81120000.

Problem 8.

There are 30000000 of inhabitants in France: in what proportion must the population increase so that there may be 81120000 at the end of a hundred and one years?

According to the formula,

$$r = -1 + \sqrt[n-1]{\frac{\omega}{a}} = -1 + \frac{L\omega - La}{n-1} =$$

$$\frac{L81120000 - L30000000}{10} = -1 + \frac{0,43200}{100} = -$$

$1 + 0,00432$; now the logarithm 0,00432 answers to the number $\frac{101}{100}$; therefore $r = -1 + \frac{101}{100} = \frac{1}{100}$. The population must therefore increase by $\frac{1}{100}$ yearly.

Annuities.

571. According to the proper import of the word, an Annuity is a yearly income. Revenues arising from Public Funds or Stocks, Donations, Pensions, Salaries, Rents, &c., the payments of some of which are usually made at shorter periods, are also called Annuities.

The claim which any one holds, in expectancy, to a revenue dependent on a future event, as the demise of one or more individuals, &c., is called an Annuity in Reversion.

572. Public Funds or Stocks are loans advanced to Government, for which interest is regularly paid, from revenues set apart for the purpose. This mode of levying taxes for the payment of interest is called the Funding System, and the loans constitute the *National Debt.*

The debts of Government differ from other contracts in this, that the public creditor can claim only his interest; he may, however, sell his stock; that is, he may transfer his claim, and thus obtain his capital, more or less, according to the price of stock, which fluctuates from a variety of causes.

573. In raising loans a *Douceur* is sometimes given by Government of an annuity for a limited time; this is called a *Terminable Annuity;* but the returns of the regular stocks on which the common interest is paid, are called *Perpetual Annuities.*

574. Peyrard, a talented French writer, to whom we are indebted for a great part of this section, says, "We call annuities equal sums, paid yearly, to extinguish in a given time a capital and its compound interest."

A person having borrowed $12000 for five years, at ten per cent, would acquit himself in five equal yearly payments: what sum must he pay at the end of each year?

It is evident that, if there were no reimbursements, he would be indebted $12000 \times (\frac{11}{10})^5$ at the end of the fifth year.

But the borrower remits a sum, b, at the end of the first year, without which b would have become $b \times (\frac{11}{10})^4$; he remits b at the end of the second year, without which b would have become $b \times (\frac{11}{10})^3$; he remits b at the end of the third year, without which b would have become $b \times (\frac{11}{10})^2$; he remits b at the end of the fourth year, without which b would have become $b \times \frac{11}{10}$; and, lastly, at the end of the fifth year, he remits b: therefore $b \times (\frac{11}{10})^4 + b \times (\frac{11}{10})^3 + b \times (\frac{11}{10})^2 + b \times (\frac{11}{10}) + b = 12000\ (\frac{11}{10})^5$.

But the numbers $b \times (\frac{11}{10})^4 + b \times (\frac{11}{10})^3 + b \times (\frac{11}{10})^2 + b \times (\frac{11}{10}) + b$ form a geometrical progression, the sum of which is equal to $\frac{b \times (\frac{11}{10})^4 \times \frac{11}{10} - b}{\frac{11}{10} - 1}$; that is to say, $\frac{b \times (\frac{11}{10})^5 - b}{\frac{1}{10}}$;

therefore $\frac{b \times (\frac{11}{10})^5 - b}{\frac{1}{10}} = 12000 \times (\frac{11}{10})^5$;

but $\frac{b \times (\frac{11}{10})^5 - b}{\frac{1}{10}} = \frac{b \times [(\frac{11}{10})^5 - 1]}{\frac{1}{10}}$;

therefore $\frac{b \times [(\frac{11}{10})^5 - 1]}{\frac{1}{10}} = 12000 \times (\frac{11}{10})^5$;

therefore $b = \frac{12000 \times (\frac{11}{10})^5 \times \frac{1}{10}}{(\frac{11}{10})^5 - 1} = \$3165\frac{34785}{61051}$ Answer.

It will be the same whatever be the sum lent, the interest, and the number of years.

Calling a the sum lent, n the number of years, r the interest, and b the sum which must be remitted at the end of each year, we shall have $b = \frac{a \times r \times (1 + r)^n}{(1 + r)^n - 1}$,

from which we deduce $a = \frac{b \times (1 + r)^n - b}{r \times (1 + r)^n}$;

$$n = \frac{\mathrm{L}[b - \mathrm{L}\,(b - a \times r)]}{\mathrm{L}\,(1 + r)}$$

With these three formulæ we can resolve all cases that Annuities present.

Let it be proposed, for example, to find what sum we must give yearly, to extinguish, in three years, a debt of \$500, the interest being at ten per cent.

Substituting these numbers in the first formula, we shall find $b = \$201 + \frac{19}{331}$. Therefore, $\$201\frac{19}{331}$, or \$201,057 + is the yearly payment.

If the interest were at 7 per cent, we should find $b =$ \$190,526 —.

575. To facilitate the calculation of Annuities, the student may construct the following Tables:

1. Find the amount of 1 unit, at every probable rate of interest, from 1 year to such a number of years as may be considered sufficient.

Thus, (568,) P being a unit, if we put r successively equal to the numbers 4, $4\frac{1}{2}$, 5, 6, we shall, by the general formula $\div\!\!\div P : P \times \frac{100 \times r}{100} : P \times \left(\frac{100 + r}{100}\right)^2 : P \times \left(\frac{100 \times r}{100}\right)^3$, &c., for the respective amounts at the end of each successive year have the numbers found in Table 1.

2. Find the amount of a deferred annuity of 1 unit from 1 year to the same number of years, and at the same rates.

We have seen (574) that the amount of an annuity b, deferred a certain number of years, is equal to the geometrical progression $b + b \times \frac{100 + r}{100} + b \times \left(\frac{100 + r}{100}\right)^2 + b \times \left(\frac{100 + r}{100}\right)^3$, &c., the index of the highest power of $\left(\frac{100 + r}{100}\right)$ being one unit less than the number of years during which the annuity has been deferred.

Now, if we suppose b a unit, the expressions $b \times \frac{100 + r}{100}$; $b \times \left(\frac{100 + r}{100}\right)^2$, &c., are the same as the compound interest of a unit for 1, 2, &c. years. Wherefore the unit b is the annuity for 1 year: $1 + 1{,}04$ (the first number in Table 1) $= 2{,}04$ is the amount of the annuity deferred 2 years; this amount, together with the second number in Table 1, is the amount for 3 years. Continuing thus to add to each result the number opposite to the same year and under the same rate in Table 1, we easily obtain the results found in Table 2.

3. Find the present worth of a Terminable Annuity of 1 unit for the same number of years, and at the same rates as those of Tables 1 and 2.

This is evidently the sum that at compound interest will equal the returns of the annuity, each, except the last, which is the annuity itself, or one unit, being placed at compound interest from the time it is due to the end of the Term. Now the amount of any capital at compound interest for any time, divided by the amount of 1 unit for the same time, will give the capital or present worth. Therefore, to form the numbers of Table 3, we divide those of Table 2, term for term, by the corresponding ones of Table 1.

Suppose, for example, that we would construct Tables 1, 2, and 3 for 10 years at the rates 4, 4½, 5, and 6 per cent. Then for Table 1. As the amount of a unit at these rates is evidently 1,04, 1,045, 1,05, and 1,06; having placed these numbers at the head of their respective columns, opposite to 1 in the column marked year, the respective amounts for 2, 3, 4, &c. years are found by raising each number successively to the 2d, 3d, 4th, &c. power, as far as the tenth inclusive.

Table 1,

Showing the Amount of one unit at Compound Interest from one Year to 10.

Year.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1.	1,040000	1,045000	1,050000	1,060000
2.	1,081600	1,092025	1,102500	1,123600
3.	1,124864	1,141166	1,157625	1,191016
4.	1,169859	1,192519	1,215506	1,262477
5.	1,216653	1,246182	1,276282	1,338226
6.	1,265319	1,302260	1,340096	1,418519
7.	1,315932	1,360862	1,407100	1,503630
8.	1,368569	1,422101	1,477455	1,593848
9.	1,423312	1,486095	1,551328	1,689479
10.	1,480244	1,552969	1,628895	1,790848

Table 2.

The Amount of an Annuity of 1 *unit from* 1 *year to* 10.

Year.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1.	1,000000	1,000000	1,000000	1,000000
2.	2,040000	2,045000	2,050000	2,060000
3.	3,121600	3,137025	3,152500	3,183600
4.	4,246464	4,278191	4,310125	4,374616
5.	5,416323	5,470710	5,525631	5,637093
6.	6,632975	6,716892	6,801913	6,975319
7.	7,898294	8,019152	8,142008	8,393838
8.	9,214226	9,380014	9,549109	9,897468
9.	10,582795	10,802114	11,026564	11,491316
10.	12,006107	12,288210	12,577893	13,180795

Table 3.

The Present Worth of an Annuity of 1 *unit from* 1 *year to* 10.

Year.	4 per cent.	4½ per cent.	5 per cent.	6 per cent.
1.	0,961538	0,956938	0,952381	0,943396
2.	1,886094	1,872668	1,859410	1,833393
3.	2,775091	2,748964	2,723248	2,673012
4.	3,629894	3,587526	3,545951	3,465105
5.	4,451822	4,389977	4,329477	4,212364
6.	5,242137	5,157872	5,075692	4,917324
7.	6,002055	5,892701	5,786373	5,582381
8.	6,732745	6,595886	6,463213	6,209793
9.	7,435332	7,268790	7,107821	6,801692
10.	8,110896	7,912718	7,721735	7,360087

576. Having constructed Tables like the preceding, as far and at such rates as we require, it is plain that if, from one of the Tables, we take the amount or present worth of a unit, for any time and at any rate, and multiply this amount or present worth by any number, we shall, for the result, obtain the amount or present worth of the number by which we multiply.

1. Suppose we would find the amount of $500 for 6 years, at 5 per cent. compound interest.

2) 1340,096 amount of 1 unit multiplied by 1000.

$670,048 Answer.

5 p. c. $= \frac{5}{100} = \frac{1}{20}$; then by continually adding $\frac{1}{20}$, we have for the successive annual amounts, 525, 551,25, 578,81, 607,75, 638,14, 670,048. Or,

$$\begin{array}{rl}
105 & \\
5 & \\
\hline
5\,25 & \text{1st year.} \\
1{,}05 & \\
\hline
551{,}25 & \text{2d “} \\
1{,}05 & \\
\hline
578{,}81\,25 & \text{3d “} \\
1{,}05 & \\
\hline
607{,}75\,31\,25 & \text{4th “} \\
1{,}05 & \\
\hline
638{,}14\,07\,81\,25 & \text{5th “} \\
1{,}05 & \\
\hline
670{,}04\,78\,20\,31\,25 & \text{6th “}
\end{array}$$

$$\begin{array}{lr}
\text{Or,} \quad \log.\ 1{,}05 \ldots\ldots\ldots & 0{,}021189 \\
& 6 \\
\hline
& 0{,}127134 \\
\log.\ 500 \ldots\ldots\ldots & 2{,}698970 \\
\hline
& 2{,}826104
\end{array}$$

This gives the answer $670,045, very nearly the same as before.

2. Required the amount of an annuity of $500, deferred 6 years, at 5 per cent.?

$$\begin{array}{l}
2)\ \underline{6801{,}913} \text{ am't of 1 unit, multiplied by 1000.} \\
\quad 3400{,}9565. \text{ Answer.}
\end{array}$$

Or, by the formula (574), as $\frac{105}{100} = \frac{21}{20}$, we have $500 + 500 \times \frac{21}{20} + 500 \times (\frac{21}{20})^2 + 500 \times (\frac{21}{20})^3 + 500 \times (\frac{21}{20})^4 + 500 \times (\frac{21}{20})^5 = 500 + 525 + 551{,}25 + 578{,}81 + 607{,}75 + 638{,}14 = \$3400{,}95$.

3. What is the present worth of an annuity of $500, deferred 6 years, at 5 per cent.?

$$\begin{array}{l}
2)\ \underline{5075{,}692} \text{ amount of } 1 \times 1000. \\
\quad 2537{,}846. \text{ Answer.}
\end{array}$$

Or, Divide the amount of the annuity by the amount of 1 unit at compound interest, thus:

1340096) 3400956500 (2537,845. Answer.
7207645
5071650
10513620
1132948
60876
7272
572

From these examples the student will perceive the very great convenience of the Tables.

577. To find the present worth of an annuity in reversion—that is, of a Terminable Annuity—to commence in a given number of years.

Rule: Find its present worth from the present time till the end of its continuance; also till the time at which it begins; the difference between these two results is the required present worth.

What is the present worth of a reversion of $500 for 6 years, to commence in 3 years from the present time, at 6 p. c.?

Observe that in taking out the Tabular Number, if, instead of the amount or present worth of 1 unit, we would have that of 10, 100, 1000, &c., we have only to remove the comma 1, 2, 3, &c. places towards the right.

6801,692 present worth of 1000 for 9 years.
2673,012 " " for 3 years.
2) 4128,680
$2064,34 required present worth.

578. To find the present worth of a perpetual annuity.

This is evidently the sum, the interest of which is equivalent to the annuity. Now as the interest of any sum is the product of the principal and the rate, considered as hundredths, (507,) we easily deduce the Rule: Divide the annuity by the rate considered as hundredths.

1. Suppose a house rents for $365; for how much should it be sold, where money is loaned at 6 per cent.?

6) 36500
$6083,33$\frac{1}{3}$ Answer.

2. The machinery in a manufactory is repaired at the end

of every 7 years at a cost of $10000; what sum will maintain the disbursement at 6 per cent.? (574.) Ans. $29855,83$\frac{5}{6}$.

Moneys of Different Countries.

Holland.

16 Pennings = 1 Stiver,
20 Stivers = 1 Florin or Guilder,
Also 12 Grotes or Pence Flemish, or 6 Stivers... = 1 Shilling Flemish,
20 Shillings, or 6 Florins = 1 Pound
2$\frac{1}{2}$ Florins, or 50 Stivers = 1 Rixdollar,
1 Florin = 40 Cents.

France.

10 Centimes (*c.*) = 1 Décime (*d.*),
10 Décimes = 1 Franc (*f.*) = 18$\frac{3}{5}$ cents.

Spain.

34 Maravedis = 1 Real of Plate,
8 Reals = 1 Dollar of Plate,
375 Maravedis = 1 Ducat of Plate or exchange,
10$\frac{5}{8}$ Reals, Plate = 1 Hard Dollar (*peso duro*) = $1 United States,
1 Real of Plate = 2 Reals Vellon.

Portugal.

1000 Rees = 1 Milree = about $1,36 in gold,
400 Rees = 1 Old Crusado, or Crusado of exch.,
480 Rees = 1 New Crusado.

Vienna.—Trieste.

4 Pfenings .... = 1 Creutzer,
60 Creutzers .... = 1 Florin or Gulden,
1$\frac{1}{2}$ Florins, or 90 Creutzers = 1 Rixdollar of account,
2 Florins, or 120 Creutzers = 1 Rixdollar, specie.

Venice.

12 Denari = 1 Soldo; 20 Soldi = 1 Lira; 6 Lire 4 Soldi piccoli = 1 Ducat current, or of account; 8 Lire piccole = 1 Ducat effective; 100 Centimes = 1 Lira Italiana; 40000 Lire piccole = 30467 lire Italiane.

Sweden.

12 Rundstycken, or ore = 1 Skilling; 48 Skillings = 1 Riksdaler.

Sicily.

20 Grani = 1 Tari, or Tarin; 30 Tari = 1 Oncia; 12 Tari = 1 Scudo, or Sicilian Crown. 5 Scudi = 2 Oncie.

Genoa.

12 Denari = 1 Soldo; 20 Soldi = 1 Lira fuori Banco; 5 Lire 15 Soldi fuori Banco = 1 Pezza. The Pezza is also divided into 20 Soldi, or 240 Denari di Pezza: thus 4 Pezze = 23 Lire; 4 Soldi di Pezza = 23 Soldi di Lira; 4 Denari di Pezza = 23 Denari di Lira.

Also, 4 Lire 12 Soldi fuori Banco = 1 Scudo di Cambio, or Crown of exchange; 10 Lire 16⅛ Soldi fuori Banco = 1 Scudo d'oro, or gold Crown; 10 Lire 14 Soldi fuori Banco = 1 Scudo d'oro marche.

Geneva.

12 Deniers = 1 Sou or Sol; 20 Sous = 1 Livre current; 3 Livres = 1 Ecu, or Patagon.

Naples.

10 Grani = 1 Carlino; 10 Carlini = 1 Ducato di regno.

Augsburg.

60 Creutzers = 1 Florin or Gulden; 90 Creutzers, or 1½ Florin = 1 Rixdollar of account; 100 Rixdollars giro, or money of exchange = 127 Rixdollars current, or 190½ Florins current; 2 Florins = 1 Rixdollar specie.

Berlin and Breslau; also Leipsic.

12 Pfenings = 1 Good Grosche; 24 Good Groschen = 1 Rixdollar currency, or of account.

Bremen.

5 Swares = 1 Grote; 32 Grotes = 1 Bremen Mark; 72 Grotes, or 2¼ Marks = 1 Rixdollar of account; 96 Grotes, or 1⅓ Rixdollars of account = 1 Rixdollar specie.

Constantinople—Smyrna.

3 Aspers = 1 Para; 40 Paras, or 120 Aspers = 1 Piastre, or Turkish Dollar; 80 Half-Paras, or 100 Minas, also called Aspers = 1 Piastre.

Copenhagen.

12 Pfenings = 1 Skilling; 16 Skillings = 1 Mark; 6 Marks Danish, or 3 Marks Lubs = 1 Ryksdaler, or Rixdollar.

Thus, 2 Pfenings, Skillings, or Marks Danish = 1 Pfening, Skilling, or Mark Lubs. (*Lubs*, money of Lubeck and Hamburg.)

Dantzic.

3 Pfenings = 1 Groschen; 30 Groschen = 1 Florin or Gulden; 8 Florins = 1 Rixdollar.

Frankfort.

4 Pfenings = 1 Creutzer; 4 Creutzers = 1 Batze; 60 Creutzers, or 15 Batzen = 1 Florin; 90 Creutzers, or 1½ Florins = 1 Rixdollar of account.

Hamburg,

Banco bears an agio on currency, which is generally from 20 to 25 per cent.

12 Pfenings = 1 Schilling, or Sol Lubs; 16 Schillings Lubs = 1 Mark; 3 Marks = 1 Rixdollar; 6 Pfenings = 1 Grote Flemish; 12 Grotes Flemish, or 6 Schillings Lubs = 1 Shilling Flemish; 20 Shillings Flemish, or 7½ Marks = 1 Pound Flemish.

Leghorn.

12 Denari di Pezza = 1 Soldo di Pezza; 20 Soldi di Pezza = 1 Pezza; commonly called Pezza of 8 Reals. 12 Denari di Lira = 1 Soldo di Lira; 20 Soldi di Lira = 1 Lira; 24 Lire moneta lunga = 23 Lire moneta buona; 6 Lire moneta lunga, or 5¾ Lire moneta buona = 1 Pezza of 8 Reals.

Milan.

12 Denari = 1 Soldo; 20 Soldi = 1 Lira; 106 Soldi or Lire imperiali = 150 Soldi or Lire correnti; 106 Soldi imperiali, or 150 Soldi correnti = 1 Filippo; 117 Soldi imperiali = 1 Scudo or Crown.

Petersburg.

10 Copecks = 1 Grieve or Grievener; 10 Grieves, or 100 Copecks = 1 Ruble.

Rome.

2 Mezzi Quattrini = 1 Quattrino; 5 Quattrini = 1 Bajocco; 10 Bajocchi = 1 Paolo; 10 Paoli, or 100 Bajocchi = 1 Scudo moneta, or Roman Crown; 12 Denari d'oro = 1 Soldo d'oro; 20 Soldi d'oro = 1 Scudo di stampa d'oro.

Turin.

12 Denari = 1 Soldo; 20 Soldi = 1 Lira.

Malta.

20 Grani = 1 Taro; 12 Tari = 1 Scudo. The Taro is likewise divided into 2 Carlini, or 120 Piccioli. $2\frac{1}{2}$ Scudi = 1 Pezza, or Dollar of exchange; 30 Tari 10 Grani = 1 Spanish Dollar; 38 Scudi 9 Tari = 1 Spanish Doubloon; 30 Tari = 1 Sicilian Dollar; 6 Scudi 3 Tari = 1 Sicilian Ounce.

Bengal, Calcutta, &c.

12 Pice = 1 Anna; 16 Annas = 1 Rupee = 41 Cents; 1 Sicca Rupee, E. I. Company = $47\frac{1}{2}$ Cents.

Bombay.

100 Rees = 1 Quarter; 4 Quarters = 1 Rupee.

Madras.

80 Cash = 1 Fanam; 45 Fanams = 1 Star Pagoda = $1,80.

China.

10 Cash = 1 Candarine; 10 Candarines = 1 Mace; 10 Mace = 1 Tale = $1,48, more or less.

Mensuration.

579. A triangle having one right angle is *right-angled;* when one angle is greater than a right angle, it is *obtuse-angled;* and when each angle is less than a right angle, it is *acute-angled.*

580. When the three sides are equal, the triangle is *equilateral;* when two sides are equal, *isosceles;* and when the sides are unequal, *scalene.*

581. As the area of any parallelogram is found in multiplying the base by its perpendicular height or altitude, (103,) it is evident that the areas of any two parallelograms have to one another the ratio which is compounded of the ratios of their bases and altitudes. Hence we deduce the following properties:

1. Parallelograms of equal base and altitude are equal to one another.
2. Parallelograms of equal base are to one another as their altitudes.
3. Parallelograms of equal altitude are to one another as their bases.

582. Because every triangle is the half of a parallelogram of equal base and altitude, it is plain (165) that the above properties also apply to triangles:

1. Triangles of equal base and altitude are equal.
2. Triangles of equal base are to one another as their altitudes.
3. Triangles of equal altitude are to one another as their bases.

583. Also to find the area of a triangle: Multiply any side by half the perpendicular drawn to it from the opposite angle. Or, multiply the perpendicular by half the side. Hence, if the triangle be right-angled, we multiply either of the two sides containing the right angle by half the other.

584. In a right-angled triangle, the square of the side subtending the right angle (which side is called the hypothenuse) is equal to the sum of the squares of the sides containing that angle. (*Euc.* 47, 1.)

Therefore, if we assume two numbers as representing two sides of a right-angled triangle, the number which represents the third side will be arbitrary; thus, if the assumed numbers are to contain the right angle, the third side or hypothenuse will be the square root of the sum of their squares; but if one of the assumed numbers is to be the hypothenuse, the third number will be the square root of the difference of the squares of the assumed numbers.

Let $x =$ hypothenuse, and a and b the other sides; then if we assume $x = 5$ and $b = 3$, we shall have

$$a = \sqrt{x^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

If $a = 6$, and $b = 8$, we shall have

$$x = \sqrt{a^2 + b^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

585. From the above it is plain that to construct a Carpenter's square, we have only to join by a pin the ends of two straight rulers, the one being placed in a horizontal and the other in a vertical position, so that they may be moved like the legs of a pair of compasses. Then if we measure 3 inches on one and 4 on the other from the angle, and move the vertical ruler so that the hypothenuse may measure exactly 5 inches, the rulers being secured in this position form the square or right angle required. Or we may take 6 inches on one ruler and 8

on the other, and move the vertical one till the hypothenuse measures exactly 10 inches. Also, we may use any other *equimultiples* or *equiparts* of the numbers 3, 4, and 5. Or, finally, find the hypothenuse corresponding to any two numbers at pleasure. The above numbers, however, being integral and easily remembered, are considered most convenient, and are, as feet, those generally employed by carpenters in framing buildings, &c.

In laying out rectangular plots of ground for gardens, fields, &c., instead of inches or feet, yards or rods may be employed. Also, the distances may be measured by a chain or other line, which can be kept uniformly tense.

586. If the circumference of a circle be divided into 6 equal arcs, the extremities of which are joined by straight lines, called chords, the figure thus inscribed in the circle is called a *hexagon;* and if from the angular points in the circumference radii be drawn, as these are all equal, as well as the chords of the arcs, the triangles so formed are equal. Now the three interior angles of every triangle (*Euc.* 32, 1) are equal to two right angles or 180 degrees, and each of the angles at the centre (302) is 60°; therefore, calling the chord the base of each triangle, the two angles at the base are equal to 120°. But the angles at the base of an isosceles triangle are equal, (*Euc.* 5, 1;) therefore, each of the angles is 60°, and the triangles are equiangular, But equiangular triangles are also equilateral, (*Euc. cor.* 6, 1;) therefore, the side of the hexagon is equal to the radius of the circle in which it is inscribed. Hence we see that the radius of a circle is the chord of an arc of 60° measured on its circumference.

587. An instrument, commonly of brass, for measuring or constructing angles, is made in the form of a quadrant or semicircle, with its perimeter or circular edge marked into the appropriate number of equal parts or degrees. This being placed with its central point at the point of the angle, and its diameter on one of the lines forming the angle, the number of degrees which is the measure of the angle is found on the perimeter, and marked or noted at the point where it is intersected by the other line, or through which the other line is to be drawn. Or, (586,) extend the compasses on the perimeter from 1° to 60°, and, having drawn a line with one foot of the compasses resting on the angular point or end of the line at which the angle is to be made, and the other beginning on the

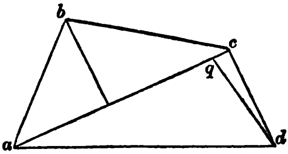
line, describe an arc of sufficient length. Then extend the compasses on the perimeter of the sector, from 1° to the required number of degrees, and set off this distance from the line on the described arc, and through the point of section and the angular point, by a ruler, draw a line which will form the required angle.

588. To measure any angle of a rectilinear field: Set a stake within the angle, equidistant from either fence, and a straight pole with a small white flag at the same distance within the other extremity of each fence, or as far as it can be conveniently seen. Then from the stake, in the direction of each pole, measure 60 links of Gunter's Chain, setting a small stick or brass rod at the farther extremity. Lastly, measure the distance in links across the angle, from one stick or rod to the other, which will give the number of degrees contained in the angle. Feet may be used instead of links.

589. From the above we can easily describe a triangle equiangular to a given triangle, namely, by constructing at the extremities of any straight line two angles equal to the two at the base of the given triangle; then because the three angles in each triangle are equal to two right angles, and if equals be taken from equals, the remainders are equal; the third angle of the one is equal to the third angle of the other. In like manner also we may construct a rectilineal figure equiangular, and consequently similar, to a given rectilineal figure; and thus the form of any piece of ground enclosed by straight lines may be plotted on paper, and its content found according to the measurement taken in the field by Gunter's Chain or otherwise. (286.)

590. The divisions of Gunter's Chain are sometimes marked upon a piece of strong tape, which is painted white on both sides, well stretched, dried, and attached to a small roller. Allowing a foot or two, next the roller, which is not marked, 22 yards are measured on the tape and neatly divided into 100 equal parts by black lines. On the other side the divisions are feet and inches. This, which is called *a tape-line*, is a neat and exceedingly useful pocket instrument.

591. To measure a four-sided field which is not rectangular: Let *a*, *b*, *c*, *d*, represent the field. Measure the distance from *a* to *c*, suppose 32 chains; then to find the perpendiculars *bg* and *dg*, place in a line with *ac* and as nearly as possible opposite to a flagstaff at *b*, a straight pole, say



10 ft. long, having a centre-piece 3 ft. long fixed at right angles, like a carpenter's square, (585,) which centre-piece may, with a plumb-line, be held vertically. Move the pole towards *a* or *c* till the centre-piece is brought in a line with *b*; mark the point *g* by setting a stick or rod, from which to *b* the perpendicular *bg* may be measured. In like manner find and mark the point *q* and measure *dq*.

Suppose *bg* = 14 chains and *dq* = 11 chains; then (583)

we have $3200 \times 700 = \overset{\text{Sq. Links.}}{2240000}$ area of *abc*.

$3200 \times 550 = \underline{1760000}$ area of triangle *adc*.

40,00000 area of field, (286.)

Hence we find that the field contains 40 acres.

Or, measure the boundary lines and angles, plot the field on paper, and measure the other distances, taken from the paper, on a scale of equal parts, from which the content may be found as before.

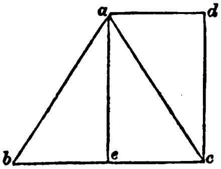
592. To construct a scale of equal parts: Draw 11 equidistant horizontal lines. Crossing these at right angles, draw any number of equidistant lines, at pleasure, according to the length of the scale, at sufficient distance from each other to allow the space between the two right-hand lines to be divided into ten equal parts. Let these be marked by 9 equidistant points on the upper and also on the lower line. Then draw ten oblique lines, the first from the top of the last but one of the perpendicular lines to the first or nearest point on the lower line; the second from the first point on the upper line to the second on the lower line, and so on to the tenth, which is drawn from the last of the 9 points on the upper line to the bottom of the last line of the scale. Then, beginning on the space next to that containing the oblique lines, number the spaces, by writing above them 1, 2, 3, &c., in succession, and the scale is complete.

The student will see that in descending on any oblique line the space from the last but one of the perpendicular lines is increased by one hundredth; therefore, to measure 5,36, we extend the compasses on the upper line, with one foot on the nearest point on the left of the figure 5 to the third of the

9 points which we marked between the two right-hand lines; this gives 5,3. Then, counting the upper line 1, we descend with both feet of the compasses to the seventh line, and find that we must extend a little farther for the 6 hundredths.

593. The three sides of a triangle being given to find the area: Find the difference between each side and half the sum of the sides; multiply the three remainders together, and this product by the half sum: the square root of the last product is the area.

What is the area of a triangle, the three sides of which measure 35 ft., 27 ft., and 19 ft.? Ans. 254,27 ft. nearly.



594. Let the straight line *bc*, be bisected—that is, equally divided, in *e*—and on *ec* let there be a right-angled parallelogram *aecd*, and join *ab* and *ac*.

Then it is plain (106) that any one of the triangles, *aeb*, *aec*, and *adc*, will coincide with each of the two others; therefore, the parallelogram is double of the triangle upon the same base *ec*, and equal to the whole triangle *abc*.

But the area of the parallelogram is equal to the product $ae \times ec$; hence the area of the triangle is found in multiplying its perpendicular height by half its base.

595. If we suppose the circumference of a circle to be cut into equal parts, and radii to be drawn to the points of section, then if we join the points of section by straight lines, we shall form triangles having their common vertex* in the centre of the circle, and so much the more numerous as there shall be a greater number of parts in the circumference. Also the bases of these triangles will approach the circumference as they increase in number, and consequently the perpendicular height of each will approach the radius. The ultimatum or limit, therefore, with regard to the sum of the bases, supposing the number infinite, is evidently the circumference itself, and hence the perpendicular height of each, the radius itself. Wherefore, the area of all the triangles—that is, the area of the circles—is (594) the product of the radius and half the circumference.

596. The exact ratio of the circumference of a circle to its

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* Angular point opposite the base.

diameter cannot, by any method hitherto discovered, be exactly expressed by numbers in the natural scale.

The approximate value of this ratio given by Archimedes was $\frac{22}{7}$, or as $3,\dot{1}4285\dot{7}$ to 1, which is too great by rather more than $\frac{1}{800}$ of the diameter.

Adrian Metius gave the ratio $\frac{355}{113}$, or as 3,141592 + to 1, which is extremely near the truth, being too great by less than half a millionth of the diameter.

Others, in modern times, have, with surprising assiduity, carried the approximation to 30, 100, and even upwards of 150 places of decimals, the importance of which the student will easily discover thus: the ratio carried to 12 places is as 3,141592653589 to 1, which is within a trillionth of the diameter. This, supposing the diameter of the globe we inhabit to be 7920 miles, gives the error in the circumference about the two-thousandth part of an inch.

The diameter of a circle being given, to find the area:

597. As the area (595) is equal to the radius multiplied by half the circumference, it is also equal to the diameter multiplied by one-fourth of the circumference, which, supposing the diameter 1, is equal to ,785398 + or nearly ,7854.

Now circles are to one another as the squares of their diameters, (*Playfair's Euc.* 6, 1, *sup.*;) therefore,

As the square of 1
To the square of the given diameter,
So is ,7854
To the required area. Hence the following Rule: Multiply the square of the given diameter by ,7854, or, if great accuracy is required, by ,785398.

What is the area of the equatorial circle, supposing its diameter 7925 miles? Ans. 49327537$\frac{7}{8}$ square miles.

598. To find the circumference of a circle from its diameter: Multiply by 3,141592.

To find the diameter from the circumference: Divide by 3,141592, or rather multiply by its reciprocal*, which is ,31831 very nearly.

599. If in a circle we draw two diameters at right angles to each other, and join their extremities, as there are four right-angles at the centre, the subtending arcs are equal and their chords equal.

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* 1 ÷ 3,141592.

Again, as the four triangles, the sides of which are radii and the base of each the chord of 90°, are right-angled and isosceles, each angle at the base is half a right angle; therefore, the angles of the inscribed figure are right angles, and, as its sides are equal, it is a square. Also the side of this square, which in each triangle is opposite to a right angle, is the square root of twice the square of the radius.

Now, the diameter being 1, the radius is ,5, and twice its square is also ,5; and extracting the root, we have $\sqrt{,5} =$,707160678, the side of the square the diameter or diagonal of which is 1.

Hence it is evident that if we draw the diagonals of a square, and take the point of intersection as a centre and the distance from this point to one of the angular points of the square as a radius, the circle will pass through the other angles of the square and be described about the square.

To find the side of the square inscribed in a given circle:

600. Seeing that, when the diameter is 1, the side of the inscribed square is ,707160678 +, and (*Euc.* 4, 6) that the sides about the equal angles of equiangular triangles are proportionals, we have the following proportion—1 : the given diameter : : ,7071607 : the side of the inscribed square.

Thus, if we find how large a square stick of timber can be hewn from a round one of 42 inches diameter at the smaller end, we have $,7071607 \times 42 = 29,7$ inches nearly. Hence the hewn stick will be 29,7 inches square.

Or thus: the diameter of the given circle is also the diagonal of the square and subtends the right angle of an isosceles triangle: therefore, its square is double of the square inscribed in the circle, and hence the following rule:

Square the diameter and take the square root of half the result. Or rather: Multiply the diameter by its half and take the square root of the result.

Thus, for the above stick of timber, we have $42 \times 21 = 882$, and $\sqrt{882} = 29,7$ inches very nearly.

1. How large a square stick of timber can be hewn from a round one, the least diameter of which is 51 inches?

Ans. $36\frac{1}{20}$ + inches.

2. The side of a square inscribed in a circle is 200 feet; what is the length of the arc which it subtends?

Ans. 222,1441 + feet.

To choose a tree out of which a stick of timber of given dimensions may be hewn:

601. Frame together 3 pieces of lath in the form of a right-angled isosceles triangle, (585,) having a plumb-line attached to one of the sides containing the right angle. Then, holding one side vertically, the other is kept pointed towards the tree we wish to cut, and withdrawn from or approached to it, as nearly as possible on a level with its base, until, by taking sight from the lower end along the hypothenuse, some part of the tree is seen exactly in range, to which point, and no farther, we know that the tree is eligible. Then, because, in equiangular triangles, sides about equal angles are proportionals, we measure from the eye to the base of the tree, which, allowance being made for cutting, is the length of the stick it will afford. Allow also, at $\frac{1}{3}$ of the height of the tree, about $\frac{1}{4}$ (in very tall forest timber $\frac{1}{5}$) diminution in the diameter of the stick, from the smaller end of which the calculation is made.

If we would have a beam 40 feet long and 20 inches square, then, supposing that we require a tree 120 ft. high, we proceed thus: $20^2 = 400$: then $400 \times 2 = 800$, and $\sqrt{800} =$ 28,284 inches, diameter of stick at the smaller end, to which we add $\frac{1}{3}$ of itself, and have $28{,}284 + 9{,}426 = 37{,}71$ inches for the diameter, and consequently, 118,47 inches for the circumference at the base. In measuring, this allowance is made for the bark.

To find the side of the inscribed square from the circumference:

602. Because the circumferences of all circles are equimultiples of their diameters, 1 : given diameter : : ,707160678 or 3,141592 : given circumference : : ,707160678 : required side. Now $,707160678 \div 3{,}141592 = {,}225096 +$. Hence the Rule: Multiply the circumference (for timber, that of smaller end) by ,225096, or, in ordinary calculation, by ,2251.

Thus, for the above stick, supposing the tree 118,47 inches at the base, subtracting $\frac{1}{4}$ of this, we have $118{,}47 - 29{,}62 =$ 88,85, and $88{,}85 \times {,}2251 = 20{,}000135$ in. for the side of the required stick.

1. A tree 150 feet high, allowance having been made for the bark, is 15 feet in circumference at the base: how large a stick 50 feet long may be hewn from it?

Ans. 32,41 + in. square.

2. A tree is 90 feet high and 7 feet in circumference at the base: how large a stick 45 feet long can be hewn from it?

Ans. 11,82 — inches.

To find the side of an equilateral triangle inscribed in a given circle:

603. If we draw the chord of double the arc subtended by the side of a hexagon, we have the side of an equilateral triangle inscribed in the circle. From one extremity of this side draw a diameter, and, by a straight line, join the other end with the other end of the diameter. The angle opposite the diameter, being in a semicircle, is (*Euc.* 31, 3) a right angle. Now one of the sides containing this angle, being the chord of $\frac{1}{6}$ of the circumference, is equal to the radius: we therefore subtract the square of the radius from the square of the diameter, and extract the square root of the result. Thus, the diameter being 1, we have

$$1^2 - {,}5^2 = 1 - {,}25 = {,}75, \text{ and } \sqrt{{,}75} = {,}8660254 +.$$

Then, because the sides about the equal angles of equiangular triangles are proportionals, 1 : diameter of any circle : : ,8660254 : the side of an equilateral triangle inscribed in that circle. Hence the rule: Multiply the diameter by ,8660254.

Required the side of an equilateral triangle inscribed in a circle of 3 feet diameter. Ans. 2,598 + feet.

To find the side of a square equal to a given circle:

604. When the diameter is 1, the area (597) is ,785398163397. The square root of this, which is, 886227, is therefore the side of an equal square.

Now, to find the square equal to any other circle, because the circles are to each other as the squares of their diameters, and also as their equal squares, the squares of the diameters (411) are to each other as the equal squares. But (414) when four squares are proportionals, their roots are proportional: therefore, 1 : diam. of any circle : : ,886227 : side of equal square. Hence the Rule: Multiply the given diameter by ,886227.

Suppose the diameter of a circle to be 7920 miles: what is the side of an equal square? Ans. 7018,92 — miles.

605. The cylinder is a solid figure generated by the revolution of a parallelogram about one of its sides, which remains fixed. Hence its bases or ends are equal circles.

To find the convex surface of a cylinder:

Multiply the circumference of one end by the length of the cylinder.

1. How many square yards of paper will cover the wall of

a circular room, the diameter of the room being 18 ft. 10 in. and its height 12 ft. 8 in.? Ans. 249$\frac{4}{5}$ +.

To find the solidity of a cylinder:

Multiply the area of one of the ends by the length.

2. What is the content of the above room in cubic yards? Ans. 130,69 +.

606. The sphere is a solid figure generated by the revolution of a semicircle about its diameter, which remains fixed.

To find the surface of a sphere:

Multiply the circumference of the circle, of which the revolving semicircle is one-half, which circle is called a great circle of the sphere, by its diameter, called also the diameter or axis of the sphere. Or multiply the square of the diameter by 3,141592.

This surface is equal to the convex surface of a cylinder, the base of which is a great circle of the sphere, and its height the diameter of the sphere.

1. What is the number of square miles of land on the surface of the globe, allowing $\frac{3}{5}$ of the whole surface to be water, the mean diameter being 7920 miles? Ans. 78824302,5715 +.

To find the solidity of a sphere:

Cube the diameter and multiply by ,5236.

2. What is the content of the earth in cubic miles? Ans. 260120860876,8.

607. Spheres are to one another as the cubes of their diameters.

1. How many balls of pure gold, one inch in diameter, can be made from one ball of the same metal, the diameter of which is 80 inches? Ans. 512000.

2. What is the comparative size of two oranges, the mean diameter of one of which is 3$\frac{1}{2}$ inches, and that of the other 5$\frac{1}{4}$ inches?

Ans. The larger orange is 3$\frac{3}{8}$ times the size of the smaller.

608. The cone is a solid figure generated by the revolution of a right-angled triangle about one of the sides containing the right angle.

To find the slant surface of a cone:

Multiply the circumference of the base by half the slant height.

1. What is the slant surface of a cone, the diameter of the base of which is 25 ft. 6 in., and the perpendicular height 75 ft. 3 in.? Ans. 3057,12 + sq. feet.

609. The pyramid is a solid, having for its base any rectilineal figure, and is such that if it be cut by a plane parallel to the base, the section presents a figure similar to the base, though smaller as it recedes from it, till, like the cone, the pyramid ends in a point, called the vertex, or summit.

To find the solidity of a pyramid or cone:

Multiply the area of the base by one third of the perpendicular height.

2. What is the solidity of the cone in the preceding example? Ans. 12810,22 — cubic feet.

3. Required the solidity of a pyramid 100 ft. high, having for its base a hexagon, the side of which is 10 feet.
Ans. 8660¼ + cubic feet.

610. When a pyramid or cone is cut by a plane parallel to the base, the upper part is still a pyramid or cone, and the lower part is called a frustrum.

To find the area of the frustrum of a cone:

To the product of the diameters of the ends add one-third of the square of the difference of those diameters. Multiply the sum by ,7854, and this product by the height. This may be applied to the measurement of round timber.

When the frustrum is a tub or other vessel, the content of which is required in wine gallons, of 231 cubic inches, with all the dimensions taken in inches, we proceed as above, except that, instead of multiplying by ,7854 and dividing by 231, we perform both these operations in simply multiplying by ,0034.

What is the content of a tub, in wine gallons, the inside diameter at the bottom being 8 feet, at the top 14 feet, and the height 10 feet? Ans. 7285,248.

To find the content of stone masonry:

611. Find the content of the wall in cubic feet and divide by 22½ for the content in perches. Or, when the wall is 1½ feet thick, multiply the length in feet by the height in feet, and divide the product by 15 for the content in perches of 22½ c. ft.

Examples.

1. What is the cost of enclosing, with a stone wall, 10 ft. high, and $1\frac{1}{2}$ ft. thick, an acre of ground, in the form of a square, at \$$2\frac{1}{2}$ per perch of $22\frac{1}{2}$ cubic feet?

Side of square, 208 ft. $8\frac{1}{2}$ in.; thickness of wall, 1 ft. 6 in.; then 210 ft. $2\frac{1}{2}$ in. $\times 4 =$ 840 ft. 10 in. $= 840\frac{5}{6}$ ft. $= \frac{5045}{6}$, whole length.

Then, by the Rule, $\frac{5045}{6} \times 10 \times 1\frac{1}{2} \div 22\frac{1}{2}$:—Or

$\frac{5045}{6} \times \frac{10}{1} \times \frac{3}{2} \times \frac{2}{45} = \frac{5045}{9} = 560,\dot{5}$ perches.

Or thus, $\frac{5045}{6} \times \frac{10}{15} = \frac{5045}{9} = 560,\dot{5}$ P., content: then $560,\dot{5} \times 2\frac{1}{2} = 140,13\dot{8} \times 10 =$ \$$1401,3\dot{8} =$ \$1401,39, the required cost.

2. Required the cost, at the same rate, of enclosing with a wall, similar to the above, a square lot containing 100 acres.

Ans. \$13914,02+.

612. Bricks are generally sold and laid by the thousand. The thickness of a brick, allowance being made for cement, is about $2\frac{1}{2}$ inches, and its width half its length, the latter being usually considered about 9 inches. This length, or rather its half, is the unit for measuring the thickness of brick walls: thus a wall is said to be $4\frac{1}{2}$ in., 9 in., 13 in., 18 in., 22 in., &c., according to the number of half-lengths of brick which it contains: hence we say that a 9-inch wall is one brick thick; a 13-inch wall, $1\frac{1}{2}$ bricks; an 18-inch wall, 2 bricks thick, &c.

To find the content of brick-work.

The number of bricks used in building a wall is found by counting the number of courses and the number of bricks in each course. Builders and jobbers, however, in making estimates, as well as in finding the content of their work, frequently use the following Rule:—Find the superficies of the wall in square feet and multiply it by 14 for every length of brick in the thickness.

Examples.

1. How many bricks will be required to build an 18-inch wall, 16 ft. high, round a garden, containing $2\frac{1}{2}$ acres, in the form of a square, and what will it cost at the rate of \$$4\frac{1}{2}$ per thousand? Ans. 594048 bricks, and the cost \$2672,216.

2. The other conditions being as above, how many bricks

will be required and what will be the cost, supposing the length of the garden equal to twice its breadth?

Ans. 629920 bricks, and the cost $2834,64.

For circular work.

613. Take the diameter of the outer circle in feet, multiply by 31416, and that product by the depth or height, taken also in feet, which will give the surface of the curb or wall in square feet. Then multiply this area by 7 for every half-length of brick in the thickness.

Examples.

1. How many bricks are required to build the wall of a circular cistern 10 ft. in diameter and 12 feet deep, the wall being 13 in.?

3,1416 × 10 × 12 = 376,992, outer surface of wall; then, 376,992 × 21 = 7916,832 or 7917, Answer.

2. How many bricks are there in a $4\frac{1}{2}$-in. curb, the diameter of which is 5 ft. and the depth 30 ft.?

Ans. 3298,68 or 3299.

A TABLE OF LOGARITHMS.

No. 1–100.									Log. 1,000000–2,000000.
No.	**Log.**	**No.**	**Log.**	**No.**	**Log.**	**No.**	**Log.**	**No.**	**Log.**
1	0,000000	21	1,322219	41	1,612784	61	1,785330	81	1,908485
2	0,301030	22	1,342423	42	1,623249	62	1,792392	82	1,913814
3	0,477121	23	1,361728	43	1,633468	63	1,799341	83	1,919078
4	0,602060	24	1,380211	44	1,643453	64	1,806180	84	1,924279
5	0,698970	25	1,397940	45	1,653213	65	1,812913	85	1,929419
6	0,778151	26	1,414973	46	1,662758	66	1,819544	86	1,934498
7	0,845098	27	1,431364	47	1,672098	67	1,826075	87	1,939519
8	0,903090	28	1,447158	48	1,681241	68	1,832509	88	1,944483
9	0,954243	29	1,462398	49	1,690196	69	1,838849	89	1,949390
10	1,000000	30	1,477121	50	1,698970	70	1,845098	90	1,954243
11	1,041393	31	1,491362	51	1,707570	71	1,851258	91	1,959041
12	1,079181	32	1,505150	52	1,716003	72	1,857332	92	1,963788
13	1,113943	33	1,518514	53	1,724276	73	1,863323	93	1,968483
14	1,146128	34	1,531479	54	1,732394	74	1,869232	94	1,973128
15	1,176091	35	1,544068	55	1,740363	75	1,875061	95	1,977724
16	1,204120	36	1,556302	56	1,748188	76	1,880814	96	1,982271
17	1,230449	37	1,568202	57	1,755875	77	1,886491	97	1,986772
18	1,255273	38	1,579784	58	1,763428	78	1,892095	98	1,991226
19	1,278754	39	1,591065	59	1,770852	79	1,897627	99	1,995635
20	1,301030	40	1,602060	60	1,778151	80	1,903090	100	2,000000

No. 1000–1600. Log. 000000–204120.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
100	000000	000434	000868	001301	001734	002166	002598	003029	003460	003891	432
101	04321	04751	05180	05609	06038	06466	06894	07321	07748	08174	428
102	08600	09026	09451	09876	10300	10724	11147	11570	11993	12415	424
103	12837	13259	13680	14100	14520	14940	15360	15779	16197	16615	420
104	17033	17451	17868	18284	18700	19116	19532	19947	20361	20775	416
105	21189	21603	22016	22428	22841	23252	23664	24075	24486	24896	412
106	25306	25715	26124	26533	26942	27350	27757	28164	28571	28978	408
107	29384	29789	30195	30600	31004	31408	31812	32216	32619	33021	404
108	33424	33826	34227	34628	35029	35430	35830	36229	36629	37028	400
109	37426	37825	38223	38620	39017	39414	39811	40207	40602	40998	397
110	41393	41787	42182	42575	42969	43362	43755	44148	44540	44931	393
111	45323	45714	46105	46495	46885	47275	47664	48053	48442	48830	390
112	49218	49606	49993	50380	50766	51152	51538	51924	52309	52694	386
113	53078	53463	53846	54230	54613	54996	55378	55760	56142	56524	383
114	56905	57286	57666	58046	58426	58805	59185	59563	59942	60320	379
115	60698	61075	61452	61829	62206	62582	62958	63333	63709	64083	376
116	64458	64832	65206	65580	65953	66326	66699	67071	67443	67814	373
117	68186	68557	68928	69298	69668	70038	70407	70776	71145	71514	370
118	71882	72250	72617	72985	73352	73718	74085	74451	74816	75182	366
119	75547	75912	76276	76640	77004	77368	77731	78094	78457	78819	363
120	79181	79543	79904	80266	80626	80987	81347	81707	82067	82426	360
121	82785	83144	83503	83861	84219	84576	84934	85291	85647	86004	357
122	86360	86716	87071	87426	87781	88136	88490	88845	89198	89552	355
123	89905	90258	90611	90963	91315	91667	92018	92370	92721	93071	352
124	93422	93772	94122	94471	94820	95169	95518	95866	96215	96562	349
125	96910	97257	97604	97951	98297	98644	98990	99335	99681	100026	346
126	100370	100715	101059	101403	101747	102090	102434	102777	103119	03462	343
127	03804	04146	04487	04828	05169	05510	05851	06191	06531	06870	341
128	07210	07549	07888	08227	08565	08903	09241	09578	09916	10253	338
129	10590	10926	11262	11598	11934	12270	12605	12940	13275	13609	335
130	13943	14277	14611	14944	15278	15610	15943	16276	16608	16940	332
131	17271	17603	17934	18265	18595	18926	19256	19586	19915	20245	330
132	20574	20903	21231	21560	21888	22216	22543	22871	23198	23525	328
133	23852	24178	24504	24830	25156	25481	25806	26131	26456	26781	325
134	27105	27429	27752	28076	28399	28722	29045	29368	29690	30012	823
135	30334	30655	30977	31298	31619	31939	32260	32580	32900	33219	321
136	33539	33858	34177	34496	34814	35133	35451	35768	36086	36403	318
137	36721	37037	37354	37670	37987	38303	38618	38934	39249	39564	316
138	39879	40194	40508	40822	41136	41450	41763	42076	42389	42702	314
139	43015	43327	43639	43951	44263	44574	44885	45196	45507	45818	311
140	46128	46438	46748	47058	47367	47676	47985	48294	48603	48911	309
141	49219	49527	49835	50142	50449	50756	51063	51370	51676	51982	307
142	52288	52594	52900	53205	53510	53815	54119	54424	54728	55032	305
143	55336	55640	55943	56246	56549	56852	57154	57457	57759	58061	303
144	58362	58664	58965	59266	59567	59868	60168	60468	60769	61068	300
145	61368	61667	61967	62266	62564	62863	63161	63460	63757	64055	298
146	64353	64650	64947	65244	65541	65838	66134	66430	66726	67022	296
147	67317	67613	67908	68203	68497	68792	69086	69380	69674	69968	294
148	70262	70555	70848	71141	71434	71726	72019	72311	72603	72895	292
149	73186	73478	73769	74060	74351	74641	74932	75222	75512	75802	290
150	76091	76381	76670	76959	77248	77536	77825	78113	78401	78689	288
151	78977	79264	79552	79839	80126	80413	80699	80986	81272	81558	287
152	81844	82129	82415	82700	82985	83270	83554	83839	84123	84407	285
153	84691	84975	85259	85542	85825	86108	86391	86674	86956	87239	283
154	87521	87803	88084	88366	88647	88928	89209	89490	89771	90051	281
155	90332	90612	90892	91171	91451	91730	92010	92289	92567	92846	279
156	93125	93403	93681	93959	94237	94514	94792	95069	95346	95623	278
157	95900	96176	96452	96729	97005	97281	97556	97832	98107	98382	276
158	98657	98932	99206	99481	99755	200029	200303	200577	200850	201124	274
159	201397	201670	201943	202216	202488	02761	03033	03305	03577	03848	272
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No. 1600–2200. Log. 204120–342423.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
160	204120	204391	204662	204933	205204	205475	205745	206016	206286	206556	271
161	06826	07095	07365	07634	07903	08172	08441	08710	08978	09247	269
162	09515	09783	10051	10318	10586	10853	11120	11388	11654	11921	267
163	12188	12454	12720	12986	13252	13518	13783	14049	14314	14579	266
164	14844	15109	15373	15638	15902	16166	16430	16694	16957	17221	264
165	17484	17747	18010	18273	18535	18798	19060	19322	19584	19846	262
166	20108	20370	20631	20892	21153	21414	21675	21936	22196	22456	261
167	22716	22976	23236	23496	23755	24015	24274	24533	24792	25051	259
168	25309	25568	25826	26084	26342	26600	26858	27115	27372	27630	258
169	27887	28144	28400	28657	28913	29170	29426	29682	29938	30193	256
170	30449	30704	30960	31215	31470	31724	31979	32233	32488	32742	255
171	32996	33250	33504	33757	34011	34264	34517	34770	35023	35276	253
172	35528	35781	36033	36285	36537	36789	37041	37292	37544	37795	252
173	38046	38297	38548	38799	39049	39299	39550	39800	40050	40300	250
174	40549	40799	41048	41297	41546	41795	42044	42293	42541	42790	249
175	43038	43286	43534	43782	44030	44277	44524	44772	45019	45266	247
176	45513	45759	46006	46252	46499	46745	46991	47236	47482	47728	246
177	47973	48219	48464	48709	48954	49198	49443	49687	49932	50176	245
178	50420	50664	50908	51151	51395	51638	51881	52125	52367	52610	243
179	52853	53096	53338	53580	53822	54064	54306	54548	54790	55031	242
180	55273	55514	55755	55996	56236	56477	56718	56958	57198	57439	241
181	57679	57918	58158	58398	58637	58877	59116	59355	59594	59833	239
182	60071	60310	60548	60787	61025	61263	61501	61738	61976	62214	238
183	62451	62688	62925	63162	63399	63636	63873	64109	64345	64582	237
184	64818	65054	65290	65525	65761	65996	66232	66467	66702	66937	235
185	67172	67406	67641	67875	68110	68344	68578	68812	69046	69279	234
186	69513	69746	69980	70213	70446	70679	70912	71144	71377	71609	233
187	71842	72074	72306	72538	72770	73001	73233	73464	73696	73927	232
188	74158	74389	74620	74850	75081	75311	75542	75772	76002	76232	230
189	76462	76691	76921	77151	77380	77609	77838	78067	78296	78525	229
190	78754	78982	79210	79439	79667	79895	80123	80351	80578	80806	228
191	81033	81261	81488	81715	81942	82169	82395	82622	82849	83075	227
192	83301	83527	83753	83979	84205	84431	84656	84882	85107	85332	226
193	85557	85782	86007	86232	86456	86681	86905	87130	87354	87578	25
194	87802	88025	88249	88473	88696	88920	89143	89366	89589	89812	223
195	90035	90257	90480	90702	90925	91147	91369	91591	91813	92034	222
196	92256	92478	92699	92920	93141	93363	93583	93804	94025	94246	221
197	94466	94687	94907	95127	95347	95567	95787	96007	96226	96446	220
198	96665	96884	97104	97323	97542	97761	97979	98198	98416	98635	219
199	98853	99071	99289	99507	99725	99943	300161	300378	300595	300813	218
200	301030	301247	301464	301681	301898	302114	02331	02547	02764	02980	217
201	03196	03412	03628	03844	04059	04275	04491	04706	04921	05136	216
202	05351	05566	05781	05996	06211	06425	06639	06854	07068	07282	214
203	07496	07710	07924	08137	08351	08564	08778	08991	09204	09417	213
204	09630	09843	10056	10268	10481	10693	10906	11118	11330	11542	212
205	11754	11966	12177	12389	12600	12812	13023	13234	13445	13656	211
206	13867	14078	14289	14499	14710	14920	15130	15340	15550	15760	210
207	15970	16180	16390	16599	16809	17018	17227	17436	17645	17854	209
208	18063	18272	18481	18689	18898	19106	19314	19522	19730	19938	208
209	20146	20354	20562	20769	20977	21184	21391	21598	21805	22012	207
210	22219	22426	22633	22839	23046	23252	23458	23665	23871	24077	206
211	24282	24488	24694	24899	25105	25310	25516	25721	25926	26131	205
212	26336	26541	26745	26950	27155	27359	27563	27767	27972	28176	204
213	28380	28583	28787	28991	29194	29398	29601	29805	30008	30211	203
214	30414	30617	30819	31022	31225	31427	31630	31832	32034	32236	202
215	32438	32640	32842	33044	33246	33447	33649	33850	34051	34253	202
216	34454	34655	34856	35056	35257	35458	35658	35859	36059	36260	201
217	36460	36660	36860	37060	37260	37459	37659	37858	38058	38257	200
218	38456	38656	38855	39054	39253	39451	39650	39849	40047	40246	199
219	40444	40642	40841	41039	41237	51435	41632	41830	42028	42225	198
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No. 2200–2800. Log. 342423–447158.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
220	342423	342620	342817	343014	343212	343409	343606	343802	343999	344196	197
221	44392	44589	44785	44981	45178	45374	45570	45766	45962	46157	196
222	46353	46549	46744	46939	47135	47330	47525	47720	47915	48110	195
223	48305	48500	48694	48889	49083	49278	49472	49666	49860	50054	194
224	50248	50442	50636	50829	51023	51216	51410	51603	51796	51989	193
225	52182	52375	52568	52761	52954	53147	53339	53532	53724	53916	193
226	54108	54301	54493	54685	54876	55068	55260	55452	55643	55834	192
227	56026	56217	56408	56599	56790	56981	57172	57363	57554	57744	191
228	57935	58125	58316	58506	58696	58886	59076	59266	59456	59646	190
229	59835	60025	60215	60404	60593	60783	60972	61161	61350	61539	189
230	61728	61917	62105	62294	62482	62671	62859	63048	63230	63424	188
231	63612	63800	63988	94176	64363	64551	64739	64926	65113	65301	188
232	65488	65675	65862	66049	66236	66423	66610	66796	66983	67169	187
233	67356	67542	67729	67915	68101	68287	68473	68659	68844	69030	186
234	69216	69401	69587	69772	69958	70143	70328	70513	70698	70883	185
235	71068	71253	71437	71622	71806	71991	72175	72360	72544	72728	184
236	72912	73096	73280	73464	73647	73831	74015	74198	74382	74565	184
237	74748	74932	75115	75298	75481	75664	75846	76029	76212	76394	183
238	76577	76759	76942	77124	77306	77488	77670	77852	78034	78216	182
239	78398	78580	78761	78943	79124	79306	79487	79668	79849	80030	181
240	80211	80392	80573	80754	80934	81115	81296	81476	81656	81837	181
241	82017	82197	82377	82557	82737	82917	83097	83277	83456	83636	180
242	83815	83995	84174	84353	84533	84712	84891	85070	85249	85428	179
243	85606	85785	85964	86142	86321	86499	86677	86856	87034	87212	178
244	87390	87568	87746	87923	88101	88279	88456	88634	88811	88989	178
245	89166	89343	89520	89697	89875	90051	90228	90405	90582	90759	177
246	90935	91112	91288	91464	91641	91817	91993	92169	92345	92521	176
247	92697	92873	93048	93224	93400	93575	93751	93926	94101	94277	176
248	94452	94627	94802	94977	95152	95326	95501	95676	95850	96025	175
249	96199	96374	96548	96722	96896	97071	97245	97418	97592	97766	174
250	97940	98114	98287	98461	98634	98808	98981	99154	99327	99501	173
251	99674	99847	400020	400192	400365	400538	400711	400883	401056	401228	173
252	401400	401573	01745	01917	02089	02261	02433	02605	02777	02949	172
253	03120	03292	03464	03635	03807	03978	04149	04320	04492	04663	171
254	04834	05005	05175	05346	05517	05688	05858	06029	06199	06370	171
255	06540	06710	06881	07051	07221	07391	07561	07731	07900	08070	170
256	08240	08410	08579	08749	08918	09087	09257	09426	09595	09764	169
257	09933	10102	10271	10440	10608	10777	10946	11114	11283	11451	169
258	11620	11788	11956	12124	12292	12460	12628	12796	12964	13132	168
259	13300	13467	13635	13802	13970	14137	14305	14472	14639	14806	167
260	14973	15140	15307	15474	15641	15808	15974	16141	16308	16474	167
261	16640	16807	16973	17139	17306	17472	17638	17804	17970	18135	166
262	18301	18467	18633	18798	18964	19129	19295	19460	19625	19791	165
263	19956	20121	20286	20451	20616	20781	20945	21110	21275	21439	165
264	21604	21768	21933	22097	22261	22426	22590	22754	22918	23082	164
265	23246	23410	23573	23737	23901	24064	24228	24392	24555	24718	164
266	24882	25045	25208	25371	25534	25697	25860	26023	26186	26349	163
267	26511	26674	26836	26999	27161	27324	27486	27648	27811	27973	162
268	28135	28297	28459	28621	28782	28944	29106	29268	29429	29591	162
269	29752	29914	30075	30236	30398	30559	30720	30881	31042	31203	161
270	31364	31525	31685	31846	32007	32167	32328	32488	32649	32809	160
271	32969	33129	33290	33450	33610	33770	33930	34090	34249	34409	160
272	34569	34728	34888	35048	35207	35366	35526	35685	35844	36003	159
273	36163	36322	36481	36640	36798	36957	37116	37275	37433	37592	159
274	37751	37909	38067	38226	38384	38542	38700	38859	39017	39175	158
275	39333	39491	39648	39806	39964	40122	40279	40437	40594	40752	158
276	40909	41066	41224	41381	41538	41695	41852	42009	42166	42323	157
277	42480	42636	42793	42950	43106	43263	43419	43576	43732	43888	157
278	44045	44201	44357	44513	44669	44825	44981	45137	45293	45448	156
279	45604	45760	45915	46071	46226	46382	46537	46692	46848	47003	155
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No. 2800–3400. Log. 447158–531479.

No	0.	1	2	3	4	5	6	7	8	9	Diff.
280	447158	447313	447468	447623	447778	447933	448088	448242	448397	448552	155
281	48706	48861	49015	49170	49324	49478	49633	49787	49941	50095	154
282	50249	50403	50557	50711	50865	51018	51172	51326	51479	51633	154
283	51786	51940	52093	52247	52400	52553	52706	52859	53012	53165	153
284	53318	53471	53624	53777	53930	54082	54235	54387	54540	54692	153
285	54845	54997	55149	55302	55454	55606	55758	55910	56062	56214	152
286	56366	56518	56670	56821	56973	57125	57276	57428	57579	57730	152
287	57882	58033	58184	58336	58487	58638	58789	58940	59091	59242	151
288	59392	59543	59694	59845	59995	60146	60296	60447	60597	60747	151
289	60898	61048	61198	61348	61498	61649	61799	61948	62098	62248	150
290	62398	62548	62697	62847	62997	63146	63296	63445	63594	63744	150
291	63893	64042	64191	64340	64489	64639	64787	64936	65085	65234	149
292	65383	65532	65680	65829	65977	66126	66274	66423	66571	66719	149
293	66868	67016	67164	67312	67460	67608	67756	67904	68052	68200	148
294	68347	68495	68643	68790	68938	69085	69233	69380	69527	69675	148
295	69822	69969	70116	70263	70410	70557	70704	70851	70998	71145	147
296	71292	71438	71585	71732	71878	72025	72171	72317	72464	72610	147
297	72756	72903	73049	73195	73341	73487	73633	73779	73925	74070	146
298	74216	74362	74508	74653	74799	74944	75090	75235	75381	75526	146
299	75671	75816	75962	76107	76252	76397	76542	76687	76832	76976	145
300	77121	77266	77411	77555	77700	77844	77989	78133	78278	78422	145
301	78566	78711	78855	78999	79143	79287	79431	79575	79719	79863	144
302	80007	80151	80294	80438	80582	80725	80869	81012	81156	81299	144
303	81443	81586	81729	81872	82016	82159	82302	82445	82588	82731	143
304	82874	83016	83159	83302	83445	83587	83730	83872	84015	84157	143
305	84300	84442	84584	84727	84869	85011	85153	85295	85437	85579	142
306	85721	85863	86005	86147	86289	86430	86572	86714	86855	86997	142
307	87138	87280	87421	87563	87704	87845	87986	88127	88269	88410	141
308	88551	88692	88833	88973	89114	89255	89396	89537	89677	89818	141
309	89958	90099	90239	90380	90520	90661	90801	90941	91081	91222	140
310	91362	91502	91642	91782	91922	92062	92201	92341	92481	92621	140
311	92760	92900	93040	93179	93319	93458	93597	93737	93876	94015	139
312	94155	94294	94433	94572	94711	94850	94989	95128	95267	95406	139
313	95544	95683	95822	95960	96099	96237	96376	96514	96653	96791	139
314	96930	97068	97206	97344	97482	97621	97759	97897	98035	98173	138
315	98311	98448	98586	98724	98862	98999	99137	99275	99412	99550	138
316	99687	99824	99962	500099	500236	500374	500511	500648	500785	500922	137
317	501059	501196	501333	01470	01607	01744	01880	02017	02154	02290	137
318	02427	02564	02700	02837	02973	03109	03246	03382	03518	03654	136
319	03791	03927	04063	04199	04335	04471	04607	04743	04878	05014	136
320	05150	05286	05421	05557	05692	05828	05963	06099	06234	06370	136
321	06505	06640	06775	06911	07046	07181	07316	07451	07586	07721	135
322	07856	07991	08125	08260	08395	08530	08664	08799	08933	09068	135
323	09202	09337	09471	09606	09740	09874	10008	10143	10277	10411	134
324	10545	10679	10813	10947	11081	11215	11348	11482	11616	11750	134
325	11883	12017	12150	12284	12417	12551	12684	12818	12951	13084	133
326	13218	13351	13484	13617	13750	13883	14016	14149	14282	14415	133
327	14548	14680	14813	14946	15079	15211	15344	15476	15609	15741	133
328	15874	16006	16139	16271	16403	16535	16668	16800	16932	17064	132
329	17196	17328	17460	17592	17724	17855	17987	18119	18251	18382	132
330	18514	18645	18777	18909	19040	19171	19303	19434	19565	19697	131
331	19828	19959	20090	20221	20352	20483	20614	20745	20876	21007	131
332	21138	21269	21400	21530	21661	21792	21922	22053	22183	22314	131
333	22444	22575	22705	22835	22966	23096	23226	23356	23486	23616	130
334	23746	23876	24006	24136	24266	24396	24526	24656	24785	24915	130
335	25045	25174	25304	25434	25563	25693	25822	25951	26081	26210	129
336	26339	26468	26598	26727	26856	26985	27114	27243	27372	27501	129
337	27630	27759	27888	28016	28145	28274	28402	28531	28660	28788	129
338	28917	29045	29174	29302	29430	29559	29687	29815	29943	30072	128
339	30200	30328	30456	30584	30712	30840	30968	31095	31223	31351	128
	0	1	2	3	4	5	6	7	8	9	

No. 3400–4000. Log. 531479–602060.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
340	531479	531607	531734	531862	531990	532117	532245	532372	532500	532627	128
341	32754	32882	33009	33136	33263	33391	33518	33645	33772	33899	127
342	34026	34153	34280	34407	34534	34661	34787	34914	35041	35167	127
343	35294	35421	35547	35674	35800	35927	36053	36179	36306	36432	126
344	36558	36685	36811	36937	37063	37189	37315	37441	37567	37693	126
345	37819	37945	38071	38197	38322	38448	38574	38699	38825	38951	126
346	39076	39202	39327	39452	39578	39703	39829	39954	40079	40204	125
347	40329	40455	40580	40705	40830	40955	41080	41205	41330	41454	125
348	41579	41704	41829	41953	42078	42203	42327	42452	42576	42701	125
349	42825	42950	43074	43199	43323	43447	43571	43696	43820	43944	124
350	44068	44192	44316	44440	44564	44688	44812	44936	45060	45183	124
351	45307	45431	45554	45678	45802	45925	46049	46172	46296	46419	124
352	46543	46666	46789	46913	47036	47159	47282	47405	47529	47652	123
353	47775	47898	48021	48144	48266	48389	48512	48635	48758	48881	123
354	49003	49126	49249	49371	49494	49616	49739	49861	49984	50106	123
355	50228	50351	50473	50595	50717	50840	50962	51084	51206	51328	122
356	51450	51572	51694	51816	51938	52059	52181	52303	52425	52546	122
357	52668	52790	52911	53033	53154	53276	53398	53519	53640	53762	121
358	53883	54004	54126	54247	54368	54489	54610	54731	54852	54973	121
359	55094	55215	55336	55457	55578	55699	55820	55940	56061	56182	121
360	56302	56423	56544	56664	56785	56905	57026	57146	57267	57387	120
361	57507	57627	57748	57867	57988	58108	58228	58348	58469	58589	120
362	58709	58829	58948	59068	59188	59308	59428	59548	59667	59787	120
363	59907	60026	60146	60265	60385	60504	60624	60743	60863	60982	119
364	61101	61221	61340	61459	61578	61698	61817	61936	62055	62174	119
365	62293	62412	62531	62650	62769	62887	63006	63125	63244	63362	119
366	63481	63600	63718	63837	63955	64074	64192	64311	64429	64548	119
367	64666	64784	64903	65021	65139	65257	65376	65494	65612	65730	118
368	65848	65966	66084	66202	66320	66437	66555.	66673	66791	66909	118
369	67026	67144	67262	67379	67497	67614	67732	67849	67967	68084	118
370	68202	68319	68436	68554	68671	68788	68905	69023	69140	69257	117
371	69374	69491	69608	69725	69842	69959	70076	70193	70309	70426	117
372	70543	70660	70776	70893	71010	71126	71243	71359	71476	71592	117
373	71709	71825	71942	72058	72174	72291	72407	72523	72639	72755	116
374	72872	72988	73104	73220	73336	73452	73568	73684	73800	73915	116
375	74031	74147	74263	74379	74494	74610	74726	74841	74957	75072	116
376	75188	75303	75419	75534	75650	75765	75880	75996	76111	76226	115
377	76341	76457	76572	76687	76802	76917	77032	77147	77262	77377	115
378	77492	77607	77721	77836	77951	78066	78181	78295	78410	78525	115
379	78639	78754	78868	78983	79097	79212	79326	79441	79555	79669	114
380	79784	79898	80012	80126	80240	80355	80469	80583	80697	80811	114
381	80925	81039	81153	81267	81381	81495	81608	81722	81836	81950	114
382	82063	82177	82291	82404	82518	82631	82745	82858	82972	83085	114
383	83199	83312	83426	83539	83652	83765	83879	83992	84105	84218	113
384	84331	84444	84557	84670	84783	84896	85009	85122	85235	85348	113
385	85461	85574	85686	85799	85912	86024	86137	86250	86362	86475	113
386	86587	86700	86812	86925	87037	87149	87262	87374	87486	87599	112
387	87711	87823	87935	88047	88160	88272	88384	88496	88608	88720	112
388	88832	88944	89056	89167	89279	89391	89503	89615	89726	89838	112
389	89950	90061	90173	90284	90396	90507	90619	90730	90842	90953	112
390	91065	91176	91287	91399	91510	91621	91732	91843	91955	92066	111
391	92177	92288	92399	92510	92621	92732	92843	92954	93064	93175	111
392	93286	93397	93508	93618	93729	93840	93950	94061	94171	94282	111
393	94393	94503	04613	94724	94834	94945	95055	95165	95276	95386	110
394	95496	95606	95717	95827	95937	96047	96157	96267	96377	96487	110
395	96597	96707	96817	96927	97037	97146	97256	97366	97476	97585	110
396	97695	97805	97914	98024	98134	98243	98353	98462	98572	98681	110
397	98790	98900	99009	99119	99228	99337	99446	99556	99665	99774	109
398	99883	99992	600101	600210	600319	600428	600537	600646	600755	600864	109
399	600973	601082	01190	01299	01408	01517	01625	01734	01843	01951	109
	0	1	2	3	4	5	6	7	8	9	

No. 4000–4600. Log. 602060–662758.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
400	602060	602169	602277	602386	602494	602603	602711	602819	602928	603036	108
401	03144	03253	03361	03469	03577	03686	03794	03902	04010	04118	108
402	04226	04334	04442	04550	04658	04766	04874	04982	05089	05197	108
403	05305	05413	05521	05628	05736	05844	05951	06059	06166	06274	108
404	06381	06489	06596	06704	06811	06919	07026	07133	07241	07348	107
405	07455	07562	07669	07777	07884	07991	08098	08205	08312	08419	107
406	08526	08633	08740	08847	08954	09061	09167	09274	09381	09488	107
407	09594	09701	09808	09914	10021	10128	10234	10341	10447	10554	107
408	10660	10767	10873	10979	11086	11192	11298	11405	11511	11617	106
409	11723	11829	11936	12042	12148	12254	12360	12466	12572	12678	106
410	12784	12890	12996	13101	13207	13313	13419	13525	13630	13736	106
411	13842	13947	14053	14159	14264	14370	14475	14581	14686	14792	106
412	14897	15003	15108	15213	15319	15424	15529	15634	15740	15845	105
413	15950	16055	16160	16265	16370	16475	16580	16685	16790	16895	105
414	17000	17105	17210	17315	17420	17524	17629	17734	17839	17943	105
415	18048	18153	18257	18362	18466	18571	18675	18780	18884	18989	105
416	19093	19198	19302	19406	19511	19615	19719	19823	19928	20032	104
417	20136	20240	20344	20448	20552	20656	20760	20864	20968	21072	104
418	21176	21280	21384	21488	21592	21695	21799	21903	22007	22110	104
419	22214	22318	22421	22525	22628	22732	22835	22939	23042	23146	104
420	23249	23353	23456	23559	23663	23766	23869	23972	24076	24179	103
421	24282	24385	24488	24591	24694	24798	24901	25004	25107	25209	103
422	25312	25415	25518	25621	25724	25827	25929	26032	26135	26238	103
423	26340	26443	26546	26648	26751	26854	26956	27058	27161	27263	103
424	27366	27468	27571	27673	27775	27878	27980	28082	28184	28287	102
425	28389	28491	28593	28695	28797	28900	29002	29104	29206	29308	102
426	29410	29511	29613	29715	29817	29919	30021	30123	30224	30326	102
427	30428	30530	30631	30733	30834	30936	31038	31139	31241	31342	102
428	31444	31545	31647	31748	31849	31951	32052	32153	32255	32356	101
429	32457	32558	32660	32761	32862	32963	33064	38165	33266	33367	101
430	33468	33569	33670	33771	33872	33973	34074	34175	34276	34376	101
431	34477	34578	34679	34779	34880	34981	35081	35182	35283	35383	101
432	35484	35584	35685	35785	35886	35986	36086	36187	36287	36388	100
433	36488	36588	36688	36789	36889	36989	37089	37189	37289	37390	100
434	37490	37590	37690	37790	37890	37990	38090	38190	38289	38389	100
435	38489	38589	38689	38789	38888	38988	39088	39188	39287	39387	100
436	39486	39586	39686	39785	39885	39984	40084	40183	40283	40382	99
437	40481	40581	40680	40779	40879	40978	41077	41176	41276	41375	99
438	41474	41573	41672	41771	41870	41970	42069	42168	42267	42366	99
439	42464	42563	42662	42761	42860	42959	43058	43156	43255	43354	99
440	43453	43551	43650	43749	43847	43946	44044	44143	44242	44340	98
441	44439	44537	44635	44734	44832	44931	45029	45127	45226	45324	98
442	45422	45520	45619	45717	45815	45913	46011	46109	46208	46306	98
443	46404	46502	46600	46698	46796	46894	46991	47089	47187	47285	98
444	47383	47481	47579	47676	47774	47872	47969	48067	48165	48262	98
445	48360	48458	48555	48653	48750	48848	48945	49043	49140	49237	97
446	49335	49432	49530	49627	49724	49821	49919	50016	50113	50210	97
447	50307	50405	50502	50599	50696	50793	50890	50987	51084	51181	97
448	51278	51375	51472	51569	51666	51762	51859	51956	52053	52150	97
449	52246	52343	52440	52536	52633	52730	52826	52923	53019	53116	97
450	53213	53309	53405	53502	53598	53695	53791	53888	53984	54080	96
451	54176	54273	54369	54465	54562	54658	54754	54850	54946	55042	96
452	55138	55234	55331	55427	55523	55619	55714	55810	55906	56002	96
453	56098	56194	56290	56386	56481	56577	56673	56769	56864	56960	96
454	57056	57151	57247	57343	57438	57534	57629	57725	57820	57916	96
455	58011	58107	58202	58298	58393	58488	58584	58679	58774	58870	95
456	58965	59060	59155	59250	59346	59441	59536	59631	59726	59821	95
457	59916	60011	60106	60201	60296	60391	60486	60581	60676	60771	95
458	60865	60960	61055	61150	61245	61339	61434	61529	61623	61718	95
459	61813	61907	62002	62096	62191	62285	62380	62474	62569	62663	95
	0	1	2	3	4	5	6	7	8	9	

No. 4600–5200. Log. 662758–716003.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
460	662758	662852	662947	663041	663135	663230	663324	663418	663512	663607	94
461	63701	63795	63889	63983	64078	64172	64266	64360	64454	64548	94
462	64642	64736	64830	64924	65018	65112	65206	65299	65393	65487	94
463	65581	65675	65769	65862	65956	66050	66143	66237	66331	66424	94
464	66518	66612	66705	66799	66892	66986	67079	67173	67266	67359	94
465	67453	67546	67640	67733	67826	67920	68013	68106	68199	68293	93
466	68386	68479	68572	68665	68758	68852	68945	69038	69131	69224	93
467	69317	69410	69503	69596	69689	69782	69875	69967	70060	70153	93
468	70246	70339	70431	70524	70617	70710	70802	70895	70988	71080	93
469	71173	71265	71358	71451	71543	71636	71728	71821	71913	72005	93
470	72098	72190	72283	72375	72467	72560	72652	72744	72836	72929	92
471	73021	73113	73205	73297	73390	73482	73574	73666	73758	73850	92
472	73942	74034	74126	74218	74310	74402	74494	74586	74677	74769	92
473	74861	74953	75045	75136	75228	75320	75412	75503	75595	75687	92
474	75778	75870	75962	76053	76145	76236	76328	76419	76511	76602	92
475	76694	76785	76876	76968	77059	77151	77242	77333	77424	77516	91
476	77607	77698	77789	77881	77972	78063	78154	78245	78336	78427	91
477	78518	78609	78700	78791	78882	78973	79064	79155	79246	79337	91
478	79428	79519	79610	79700	79791	79882	79973	80063	80154	80245	91
479	80335	80426	80517	80607	80698	80789	80879	80970	81060	81151	91
480	81241	81332	81422	81513	81603	81693	81784	81874	81964	82055	90
481	82145	82235	82326	82416	82506	82596	82686	82777	82867	82957	90
482	83047	83137	83227	83317	83407	83497	83587	83677	83767	83857	90
483	83947	84037	84127	84217	84307	84396	84486	84576	84666	84756	90
484	84845	84935	85025	85114	85204	85294	85383	85473	85563	85652	90
485	85742	85831	85921	86010	86100	86189	86279	86368	86457	86547	89
486	86636	86726	86815	86904	86994	87083	87172	87261	87351	87440	89
487	87529	87618	87707	87796	87886	87975	88064	88153	88242	88331	89
488	88420	88509	88598	88687	88776	88865	88953	89042	89131	80220	89
489	89309	89398	89486	89575	89664	89753	89841	89930	90019	90107	89
490	90196	90285	90373	90462	90550	90639	90727	90816	90905	90993	89
491	91081	91170	91258	91347	91435	91524	91612	91700	91789	91877	88
492	91965	92053	92142	92230	92318	92406	92494	92583	92671	92759	88
493	92847	92935	93023	93111	93199	93287	93375	93463	93551	93639	88
494	93727	93815	93903	93991	94078	94166	94254	94342	94430	94517	88
495	94605	94693	94781	94868	94956	95044	95131	95219	95307	95394	88
496	95482	95569	95657	95744	95832	95919	96007	96094	96182	96269	87
497	96356	96444	96531	96618	96706	96793	96880	96968	97055	97142	87
498	97229	97317	97404	97491	97578	97665	97752	97839	97926	98013	87
499	98100	98188	98275	98362	98448	98535	98622	98709	98796	98883	87
500	98970	99057	99144	99231	99317	99404	99491	99578	99664	99751	87
501	99838	99924	700011	700098	700184	700271	700358	700444	700531	700617	87
502	700704	700790	00877	00963	01050	01136	01222	01309	01395	01482	86
503	01568	01654	01741	01827	01913	01999	02086	02172	02258	02344	86
504	02430	02517	02603	02689	02775	02861	02947	03033	03119	03205	86
505	03291	03377	03463	03549	03635	03721	03807	03893	03979	04065	86
506	04150	04236	04322	04408	04494	04579	04665	04751	04837	04922	86
507	05008	05094	05179	05265	05350	05436	05522	05607	05693	05778	86
508	05864	05949	06035	06120	06206	06291	06376	06462	06547	06632	85
509	06718	06803	06888	06974	07059	07144	07229	07315	07400	07485	85
510	07570	07655	07740	07826	07911	07996	08081	08166	08251	08336	85
511	08421	08506	08591	08676	08761	08846	08931	09015	09100	09185	85
512	09270	09355	09440	09524	09609	09694	09779	09863	09948	10033	85
513	10117	10202	10287	10371	10456	10540	10625	10710	10794	10879	85
514	10963	11048	11132	11216	11301	11385	11470	11554	11638	11723	84
515	11807	11892	11976	12060	12144	12229	12313	12397	12481	12566	84
516	12650	12734	12818	12902	12986	13070	13154	13238	13322	13406	84
517	13490	13574	13658	13742	13826	13910	13994	14078	14162	14246	84
518	14330	14414	14497	14581	14665	14749	14832	14916	15000	15084	84
519	15167	15251	15335	15418	15502	15586	15669	15753	15836	15920	84
	0	1	2	3	4	5	6	7	8	9	

No. 5200–5800. Log. 716003–763428.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
520	716003	716087	716170	716254	716337	716421	716504	716588	716671	716754	83
521	16838	16921	17004	17088	17171	17254	17338	17421	17504	17587	83
522	17671	17754	17837	17920	18003	18086	18169	18253	18336	18419	83
523	18502	18585	18668	18751	18834	18917	19000	19083	19165	19248	83
524	19331	19414	19497	19580	19663	19745	19828	19911	19994	20077	83
525	20159	20242	20325	20407	20490	20573	20655	20738	20821	20903	83
526	20986	21068	21151	21233	21316	21398	21481	21563	21646	21728	82
527	21811	21893	21975	22058	22140	22222	22305	22387	22469	22552	82
528	22634	22716	22798	22881	22963	23045	23127	23209	23291	23374	82
529	23456	23538	23620	23702	23784	23866	23948	24030	24112	24194	82
530	24276	24358	24440	24522	24603	24685	24767	24849	24931	25013	82
531	25095	25176	25258	25340	25422	25503	25585	25667	25748	25830	82
532	25912	25993	26075	26156	26238	26320	26401	26483	26564	26646	82
533	26727	26809	26890	26972	27053	27134	27216	27297	27379	27460	81
534	27541	27623	27704	27785	27866	27948	28029	28110	28191	28273	81
535	28354	28435	28516	28597	28678	28759	28841	28922	29003	29084	81
536	29165	29246	29327	29408	29489	29570	29651	29732	29813	29893	81
537	29974	30055	30136	30217	30298	30378	30459	30540	30621	30702	81
538	30782	30863	30944	31024	31105	31186	31266	31347	31428	31508	81
539	31589	31669	31750	31830	31911	31991	32072	32152	32233	32313	81
540	32394	32474	32555	32635	32715	32796	32876	32956	33037	33117	80
541	33197	33278	33358	33438	33518	33598	33679	33759	33839	33919	80
542	33999	34079	34159	34240	34320	34400	34480	34560	34640	34720	80
543	34800	34880	34960	35040	35120	35200	35279	35359	35439	35519	80
544	35599	35679	35759	35838	35918	35998	36078	36157	36237	36317	80
545	36396	36476	36556	36635	36715	36795	36874	36954	37034	37113	80
546	87193	37272	37352	37431	37511	37590	37670	37749	37829	37908	79
547	37987	38067	38146	38225	38305	38384	38463	38543	38622	38701	79
548	38781	38860	38939	39018	39097	39177	39256	39335	39414	39493	79
549	39572	39651	39730	39810	39889	39968	40047	40126	40205	40284	79
550	40363	40442	40521	40599	40678	40757	40836	40915	40994	41073	79
551	41152	41230	41309	41388	41467	41546	41624	41703	41782	41860	79
552	41939	42018	42096	42175	42254	42332	42411	42489	42568	42647	79
553	42725	42804	42882	42961	43039	43118	43196	43275	43353	43431	78
554	43510	43588	43667	43745	43823	43902	43980	44058	44136	44215	78
555	44293	44371	44449	44528	44606	44684	44762	44840	44919	44997	78
556	45075	45153	45231	45309	45387	45465	45543	45621	45699	45777	78
557	45855	45933	46011	46089	46167	46245	46323	46401	46479	46556	78
558	46634	46712	46790	46868	46945	47023	47101	47179	47256	47334	78
559	47412	47489	47567	47645	47722	47800	47878	47955	48033	48110	78
560	48188	48266	48343	48421	48498	48576	48653	48731	48808	48885	77
561	48963	49040	49118	49195	49272	49350	49427	49504	49582	49659	77
562	49736	49814	49891	49968	50045	50123	50200	50277	50354	50431	77
563	50508	50586	50663	50740	50817	50894	50971	51048	51125	51202	77
564	51279	51356	51433	51510	51587	51664	51741	51818	51895	51972	77
565	52048	52125	52202	52279	52356	52433	52509	52586	52663	52740	77
566	52816	52893	52970	53047	53123	53200	53277	53353	53430	53506	77
567	53583	53660	53736	53813	53889	53966	54042	54119	54195	54272	77
568	54348	54425	54501	54578	54654	54730	54807	54883	54960	55036	76
569	55112	55189	55265	55341	55417	55494	55570	55646	55722	55799	76
570	55875	55951	56027	56103	56180	56256	56332	56408	56484	56560	76
571	56636	56712	56788	56864	56940	57016	57092	57168	57244	57320	76
572	57396	57472	57548	57624	57700	57775	57851	57927	58003	58079	76
573	58155	58230	58306	58382	58458	58533	58609	58685	58761	58836	76
574	58912	58988	59063	59139	59214	59290	59366	59441	59517	59592	76
575	59668	59743	59819	59894	59970	60045	60121	60196	60272	60347	75
576	60422	60498	60573	60649	60724	60799	60875	60950	61025	61101	75
577	61176	61251	61326	61402	61477	61552	61627	61702	61778	61853	75
578	61928	62003	62078	62153	62228	62303	62378	62453	62529	62604	75
579	62679	62754	62829	62904	62978	63053	63128	63203	63278	63353	75
	0	1	2	3	4	5	6	7	8	9	

No. 5800–6400. Log. 763428–806180.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
580	763428	763503	763578	763653	763727	763802	763877	763952	764027	764101	75
581	64176	64251	64326	64400	64475	64550	64624	64699	64774	64848	75
582	64923	64998	65072	65147	65221	65296	65370	65445	65520	65594	75
583	65669	65743	65818	65892	65966	66041	66115	66190	66264	66338	74
584	66413	66487	66562	66636	66710	66785	66859	66933	67007	67082	74
585	67156	67230	67304	67379	67453	67527	67601	67675	67749	67823	74
586	67898	67972	68046	68120	68194	68268	68342	68416	68490	68564	74
587	68638	68712	68786	68860	68934	69008	69082	69156	69230	69303	74
588	69377	69451	69525	69599	69673	69746	69820	69894	69968	70042	74
589	70115	70189	70263	70336	70410	70484	70557	70631	70705	70778	74
590	70852	70926	70999	71073	71146	71220	71293	71367	71440	71514	74
591	71587	71661	71734	71808	71881	71955	72028	72102	72175	72248	73
592	72322	72395	72468	72542	72615	72688	72762	72835	72908	72981	73
593	73055	73128	73201	73274	73348	73421	73494	73567	73640	73713	73
594	73786	73860	73933	74006	74079	74152	74225	74298	74371	74444	73
595	74517	74590	74663	74736	74809	74882	74955	75028	75100	75173	73
596	75246	75319	75392	75465	75538	75610	75683	75756	75829	75902	73
597	75974	76047	76120	76193	76265	76338	76411	76483	76556	76629	73
598	76701	76774	76846	76919	76992	77064	77137	77209	77282	77354	73
599	77427	77499	77572	77644	77717	77789	77862	77934	78006	78079	73
600	78151	78224	78296	78368	78441	78513	78585	78658	78730	78802	72
601	78874	78947	79019	79091	79163	79236	79308	79380	79452	79524	72
602	79596	79669	79741	79813	79885	79957	80029	80101	80173	80245	72
603	80317	80389	80461	80533	80605	80677	80749	80821	80893	80965	72
604	81037	81109	81181	81253	81324	81396	81468	81540	81612	81684	72
605	81755	81827	81899	81971	82042	82114	82186	82258	82329	82401	72
606	82473	82544	82616	82688	82759	82831	82902	82974	83046	83117	72
607	83189	83260	83332	83403	83475	83546	83618	83689	83761	83832	71
608	83904	83975	84046	84118	84189	84261	84332	84403	84475	84546	71
609	84617	84689	84760	84831	84902	84974	85045	85116	85187	85259	71
610	85330	85401	85472	85543	85615	85686	85757	85828	85899	85970	71
611	86041	86112	86183	86254	86325	86396	86467	86538	86609	86680	71
612	86751	86822	86893	86964	87035	87106	87177	87248	87319	87390	71
613	87460	87531	87602	87673	87744	87815	87885	87956	88027	88098	71
614	88168	88239	88310	88381	88451	88522	88593	88663	88734	88804	71
615	88875	88946	89016	89087	89157	89228	89299	89369	89440	89510	71
616	89581	89651	89722	89792	89863	89933	90004	90074	90144	90215	70
617	90285	90356	90426	90496	90567	90637	90707	90778	90848	90918	70
618	90988	91059	91129	91199	91269	91340	91410	91480	91550	91620	70
619	91691	91761	91831	91901	91971	92041	92111	92181	92252	92322	70
620	92392	92462	92532	92602	92672	92742	92812	92882	92952	93022	70
621	93092	93162	93231	93301	93371	93441	93511	93581	93651	93721	70
622	93790	93860	93930	94000	94070	94139	94209	94279	94349	94418	70
623	94488	94558	94627	94697	94767	94836	94906	94976	95045	95115	70
624	95185	95254	95324	95393	95463	95532	95602	95671	95741	95811	69
625	95880	95949	96019	96088	96158	96227	96297	96366	96436	96505	69
626	96574	96644	96713	96782	96852	96921	96990	97060	97129	97198	69
627	97268	97337	97406	97475	97545	97614	97683	97752	97821	97890	69
628	97960	98029	98098	98167	98236	98305	98374	98443	98512	98582	69
629	98651	98720	98789	98858	98927	98996	99065	99134	99203	99272	69
630	99341	99409	99478	99547	99616	99685	99754	99823	99892	99961	69
631	800029	800098	800167	800236	800305	800373	800442	800511	800580	800648	69
632	00717	00786	00854	00923	00992	01060	01129	01198	01266	01335	69
633	01404	01472	01541	01609	01678	01747	01815	01884	01952	02021	69
634	02089	02158	02226	02295	02363	02432	02500	02568	02637	02705	68
635	02774	02842	02910	02979	03047	03116	03184	03252	03321	03389	68
636	03457	03525	03594	03662	03730	03798	03867	03935	04003	04071	68
637	04139	04208	04276	04344	04412	04480	04548	04616	04685	04753	68
638	04821	04889	04957	05025	05093	05161	05229	05297	05365	05433	68
639	05501	05569	05637	05705	05773	05841	05908	05976	06044	06112	68
	0	1	2	3	4	5	6	7	8	9	

No. 6400–7000. Log. 806180–845098.

No	0	1	2	3	4	5	6	7	8	9	Diff.
640	806180	806248	806316	806384	806451	806519	806587	806655	806723	806790	68
641	06858	06926	06994	07061	07129	07197	07264	07332	07400	07467	68
642	07535	07603	07670	07738	07806	07873	07941	08008	08076	08143	68
643	08211	08279	08346	08414	08481	08549	08616	08684	08751	08818	68
644	08886	08953	09021	09088	09156	09223	09290	09358	09425	09492	67
645	09560	09627	09694	09762	09829	09896	09964	10031	10098	10165	67
646	10233	10300	10367	10434	10501	10569	10636	10703	10770	10837	67
647	10904	10971	11038	11106	11173	11240	11307	11374	11441	11508	67
648	11575	11642	11709	11776	11843	11910	11977	12044	12111	12178	67
649	12245	12312	12378	12445	12512	12579	12646	12713	12780	12847	67
650	12913	12980	13047	13114	13181	13247	13314	13381	13448	13514	67
651	13581	13648	13714	13781	13848	13914	13981	14048	14114	14181	67
652	14248	14314	14381	14447	14514	14581	14647	14714	14780	14847	67
653	14913	14980	15046	15113	15179	15246	15312	15378	15445	15511	66
654	15578	15644	15711	15777	15843	15910	5976	16042	16109	16175	66
655	16241	16308	16374	16440	16506	16573	16639	16705	16771	16838	66
656	16904	16970	17036	17102	17169	17235	17301	17367	17433	17499	66
657	17565	17631	17698	17764	17830	17896	17962	18028	18094	18160	66
658	18226	18292	18358	18424	18490	18556	18622	18688	18754	18819	66
659	18885	18951	19017	19083	19149	19215	19281	19346	19412	19478	66
660	19544	19610	19675	19741	19807	19873	19939	20004	20070	20136	66
661	20201	20267	20333	20399	20464	20530	20595	20661	20727	20792	66
662	20858	20924	20989	21055	21120	21186	21251	21317	21382	21448	66
663	21514	21579	21644	21710	21775	21841	21906	21972	22037	22103	65
664	22168	22233	22299	22364	22430	22495	22560	22626	22691	22756	65
665	22822	22887	22952	23018	23083	23143	23213	23279	23344	23409	65
666	23474	23539	23605	23670	23735	23800	23865	23930	23996	24061	65
667	24126	24191	24256	24321	24386	24451	24516	24581	24646	24711	65
668	24776	24841	24906	24971	25036	25101	25166	25231	25296	25361	65
669	25426	25491	25556	25621	25686	25751	25815	25880	25945	26010	65
670	26075	26140	26204	26269	26334	26399	26464	26528	26593	26658	65
671	26723	26787	26852	26917	26981	27046	27111	27175	27240	27305	65
672	27369	27434	27498	27563	27628	27692	27757	27821	27886	27951	65
673	28015	28080	28144	28209	28273	28338	28402	28467	28531	28595	64
674	28660	28724	28789	28853	28918	28982	29046	29111	29175	29239	64
675	29304	29368	29432	29497	29561	29625	29690	29754	29818	29882	64
676	29947	30011	30075	30139	30204	30268	30332	30396	30460	30525	64
677	30589	30653	30717	30781	30845	30909	30973	31037	31102	31166	64
678	31230	31294	31358	31422	31486	31550	31614	31678	31742	31806	64
679	31870	31934	31998	32062	32126	32189	32253	32317	32381	32445	64
680	32509	32573	32637	32700	32764	32828	32892	32956	33020	33083	64
681	33147	33211	33275	33338	33402	33466	33530	33593	33657	33721	64
682	33784	33848	33912	33975	34039	34103	34166	34230	34293	34357	64
683	34421	34484	34548	34611	34675	34739	34802	34866	34929	34993	64
684	35056	35120	35183	35247	35310	35373	35437	35500	35564	35627	63
685	35691	35754	35817	35881	35944	36007	36071	36134	36197	36261	63
686	36324	36387	36451	36514	36577	36641	36704	36767	36830	36894	63
687	36957	37020	37083	37146	37210	37273	37336	37399	37462	37525	63
688	37588	37652	37715	37778	37841	37904	37967	38030	38093	38156	63
689	38219	38282	38345	38408	38471	38534	38597	38660	38723	38786	63
690	38849	38912	38975	39038	39101	39164	39227	39289	39352	39415	63
691	39478	39541	39604	39667	39729	39792	39855	39918	39981	40043	63
692	40106	40169	40232	40294	40357	40420	40482	40545	40608	40671	63
693	40733	40796	40859	40921	40984	41046	41109	41172	41234	41297	63
694	41359	41422	41485	41547	41610	41672	41735	41797	41860	41922	63
695	41985	42047	42110	42172	42235	42297	42360	42422	42484	42547	62
696	42609	42672	42734	42796	43859	42921	42983	43046	43108	43170	62
697	43233	43295	43357	43420	43482	43544	43606	43669	43731	43793	62
698	43855	43918	43980	44042	44104	44166	44229	44291	44353	44415	62
699	44477	44539	44601	44664	44726	44788	44850	44912	44974	45036	62
	0	1	2	3	4	5	6	7	8	9	

No. 7000–7600. Log. 845098–880814.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
700	845098	845160	845222	845284	845346	845408	845470	845532	845594	845656	62
701	45718	45780	45842	45904	45966	46028	46090	46151	46213	46275	62
702	46337	46399	46461	46523	46584	46646	46708	46770	46832	46894	62
703	46955	47017	47079	47141	47202	47264	47326	47388	47449	47511	62
704	47573	47634	47696	47758	47819	47881	47943	48004	48066	48127	62
705	48189	48251	48312	48374	48435	48497	48559	48620	48682	48743	62
706	48805	48866	48928	48989	49051	49112	49174	49235	49296	49358	61
707	49419	49481	49542	49604	49665	49726	49788	49849	49911	49972	61
708	50033	50095	50156	50217	50279	50340	50401	50462	50524	50585	61
709	50646	50707	50769	50830	50891	50952	51014	51075	51136	51197	61
710	51258	51320	51381	51442	51503	51564	51625	51686	51747	51808	61
711	51870	51931	51992	52053	52114	52175	52236	52297	52358	52419	61
712	52480	52541	52602	52663	52724	52785	52846	52907	52968	53029	61
713	53090	53150	53211	53272	53333	53394	53455	53516	53576	53637	61
714	53698	53759	53820	53881	53941	54002	54063	54124	54185	54245	61
715	54306	54367	54427	54488	54549	54610	54670	54731	54792	54852	61
716	54913	54974	55034	55095	55156	55216	55277	55337	55398	55459	61
717	55519	55580	55640	55701	55761	55822	55882	55943	56003	56064	61
718	56124	56185	56245	56306	56366	56427	56487	56548	56608	56668	60
719	56729	56789	56850	56910	56970	57031	57091	57151	57212	57272	60
720	57332	57393	57453	57513	57574	57634	57694	57754	57815	57875	60
721	57935	57995	58056	58116	58176	58236	58297	58357	58417	58477	60
722	58537	58597	58657	58718	58778	58838	58898	58958	59018	59078	60
723	59138	59198	59258	59318	59378	59438	59499	59559	59619	59679	60
724	59739	59799	59858	59918	59978	60038	60098	60158	60218	60278	60
725	60338	60398	60458	60518	60578	60637	60697	60757	60817	60877	60
726	60937	60996	61056	61116	61176	61236	61295	61355	61415	61475	60
727	61534	61594	61654	61714	61773	61833	61893	61952	62012	62072	60
728	62131	62191	62251	62310	62370	62430	62489	62549	62608	62668	60
729	62728	62787	62847	62906	62966	63025	63085	63144	63204	63263	60
730	63323	63382	63442	63501	63561	63620	63680	63739	63798	63858	59
731	63917	63977	64036	64096	64155	64214	64274	64333	64392	64452	59
732	64511	64570	64630	64689	64748	64808	64867	64926	64985	65045	59
733	65104	65163	65222	65282	65341	65400	65459	65518	65578	65637	59
734	65696	65755	65814	65874	65933	65992	66051	66110	66169	66228	59
735	66287	66346	66405	66465	66524	66583	66642	66701	66760	66819	59
736	66878	66937	66996	67055	67114	67173	67232	67291	67350	67409	59
737	67467	67526	67585	67644	67703	67762	67821	67880	67939	67998	59
738	68056	68115	68174	68233	68292	68350	68409	68468	68527	68586	59
739	68644	68703	68762	68821	68879	68938	68997	69056	69114	69173	59
740	69232	69290	69349	69408	69466	69525	69584	69642	69701	69760	59
741	69818	69877	69935	69994	70053	70111	70170	70228	70287	70345	59
742	70404	70462	70521	70579	70638	70696	70755	70813	70872	70930	58
743	70989	71047	71106	71164	71223	71281	71339	71398	71456	71515	58
744	71573	71631	71690	71748	71806	71865	71923	71981	72040	72098	58
745	72156	72215	72273	72331	72389	72448	72506	72564	72622	72681	58
746	72739	72797	72855	72913	72972	73030	73088	73146	73204	73262	58
747	73321	73379	73437	73495	73553	73611	73669	73727	73785	73848	58
748	73902	73960	74018	74076	74134	74192	74250	74308	74366	74424	58
749	74482	74540	74598	74656	74714	74772	74830	74887	74945	75003	58
750	75061	75119	75177	75235	75293	75351	75409	75466	75524	75587	58
751	75640	75698	75756	75813	75871	75929	75987	76045	76102	76160	58
752	76218	76276	76333	76391	76449	76507	76564	76622	76680	76737	58
753	76795	76853	76910	76968	77026	77083	77141	77198	77256	77314	58
754	77371	77429	77486	77544	77602	77659	77717	77774	77832	77889	58
755	77947	78004	78062	78119	78177	78234	78292	78349	78407	78464	57
756	78522	78579	78637	78694	78751	78809	78866	78924	78981	79038	57
757	79096	79153	79211	79268	79325	79383	79440	79497	79555	79612	57
758	79669	79726	79784	79841	79898	79956	80013	80070	80127	80185	57
759	80242	80299	80356	80413	80471	80528	80585	80642	80699	80756	57
	0	1	2	3	4	5	6	7	8	9	

No. 7600–8200. Log. 880814–913814.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
760	880814	880871	880928	880985	881042	881099	881156	881213	881270	881328	57
761	81385	81442	81499	81556	81613	81670	81727	81784	81841	81898	57
762	81955	82012	82069	82126	82183	82240	82297	82354	82411	82468	57
763	82524	82581	82638	82695	82752	82809	82866	82923	82980	83037	57
764	83093	83150	83207	83264	83321	83377	83434	83491	83548	83605	57
765	83661	83718	83775	83832	83888	83945	84002	84059	84115	84172	57
766	84229	84285	84342	84399	84455	84512	84569	84625	84682	84739	57
767	84795	84852	84909	84965	85022	85078	85135	85192	85248	85305	57
768	85361	85418	85474	85531	85587	85644	85700	85757	85813	85870	57
769	85926	85983	86039	86096	86152	86209	86265	86321	86378	86434	56
770	86491	86547	86604	86660	86716	86773	86829	86885	86942	86998	56
771	87054	87111	87167	87223	87280	87336	87392	87449	87505	87561	56
772	87617	87674	87730	87786	87842	87898	87955	88011	88067	88123	56
773	88179	88236	88292	88348	88404	88460	88516	88573	88629	88685	56
774	88741	88797	88853	88909	88965	89021	89077	89134	89190	89246	56
775	89302	89358	89414	89470	89526	89582	89638	89694	89750	89806	56
776	89862	89918	89974	90030	90086	90141	90197	90253	90309	90365	56
777	90421	90477	90533	90589	90644	90700	90756	90812	90868	90924	56
778	90980	91035	91091	91147	91203	91259	91314	91370	91426	91482	56
779	91537	91593	91649	91705	91760	91816	91872	91928	91983	92039	56
780	92095	92150	92206	92262	92317	92373	92429	92484	92540	92595	56
781	92651	92707	92762	92818	92873	92929	92985	93040	93096	93151	56
782	93207	93262	93318	93373	93429	93484	93540	93595	93651	93706	56
783	93762	93817	93873	93928	93984	94039	94094	94150	94205	94261	55
784	94316	94371	94427	94482	94538	94593	94648	94704	94759	94814	55
785	94870	94925	94980	95036	95091	95146	95201	95257	95312	95367	55
786	95423	95478	95533	95588	95643	95699	95754	95809	95864	95920	55
787	95975	96030	96085	96140	96195	96251	96306	96361	96416	96471	55
788	96526	96581	96636	96692	96747	96802	96857	96912	96967	97022	55
789	97077	97132	97187	97242	97297	97352	97407	97462	97517	97572	55
790	97627	97682	97737	97792	97847	97902	97957	98012	98067	98122	55
791	98176	98231	98286	98341	98396	98451	98506	98561	98615	98670	55
792	98725	98780	98835	98890	98944	98999	99054	99109	99164	99218	55
793	99273	99328	99383	99437	99492	99547	99602	99656	99711	99766	55
794	99820	99875	99930	99985	900039	900094	900149	900203	900258	900312	55
795	900367	900422	900476	900531	00586	00640	00695	00749	00804	00858	55
796	00913	00968	01022	01077	01131	01186	01240	01295	01349	01404	55
797	01458	01513	01567	01622	01676	01731	01785	01840	01894	01948	54
798	02003	02057	02112	02166	02221	02275	02329	02384	02438	02491	54
799	02547	02601	02655	02710	02764	02818	02873	02927	02981	03036	54
800	03090	03144	03198	03253	03307	03361	03416	03470	03524	03578	54
801	03632	03687	03741	03795	03849	03903	03958	04012	04066	04120	54
802	04174	04228	04283	04337	04391	04445	04499	04553	04607	04661	54
803	04715	04770	04824	04878	04932	04986	05040	05094	05148	05202	54
804	05256	05310	05364	05418	05472	05526	05580	05634	05688	05742	54
805	05796	05850	05904	05958	06012	06065	06119	06173	06227	06281	54
806	06335	06389	06443	06497	06550	06604	06658	06712	06766	06820	54
807	06873	06927	06981	07035	07089	07142	07196	07250	07304	07358	54
808	07411	07465	07519	07573	07626	07680	07734	07787	07841	07895	54
809	07948	08002	08056	08109	08163	08217	08270	08324	08378	08431	54
810	08485	08539	08592	08646	08699	08753	08807	08860	08914	08967	54
811	09021	09074	09128	09181	09235	09288	09342	09395	09449	09502	54
812	09556	09609	09663	09716	09770	09823	09877	09930	09984	10037	53
813	10090	10144	10197	10251	10304	10358	10411	10464	10518	10571	53
814	10624	10678	10731	10784	10838	10891	10944	10998	11051	11104	53
815	11158	11211	11264	11317	11371	11424	11477	11530	11584	11637	53
816	11690	11743	11797	11850	11903	11956	12009	12063	12116	12169	53
817	12222	12275	12328	12381	12435	12488	12541	12594	12647	12700	53
818	12753	12806	12859	12913	12966	13019	13072	13125	13178	13231	53
819	13284	13337	13390	13443	13496	13549	13602	13655	13708	13761	53
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No. 8200–8800. Log. 913814–944483.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
820	913814	913867	913920	913973	914026	914079	914131	914184	914237	914290	53
821	14343	14396	14449	14502	14555	14608	14660	14713	14766	14819	53
822	14872	14925	14977	15030	15083	15136	15189	15241	15294	15347	53
823	15400	15453	15505	15558	15611	15664	15716	15769	15822	15874	53
824	15927	15980	16033	16085	16138	16191	16243	16296	16349	16401	53
825	16454	16507	16559	16612	16664	16717	16770	16822	16875	16927	53
826	16980	17033	17085	17138	17190	17243	17295	17348	17400	17453	53
827	17505	17558	17610	17663	17715	17768	17820	17873	17925	17978	52
828	18030	18083	18135	18188	18240	18292	18345	18397	18450	18502	52
829	18555	18607	18659	18712	18764	18816	18869	18921	18973	19026	52
830	19078	19130	19183	19235	19287	19340	19392	19444	19496	19549	52
831	19601	19653	19705	19758	19810	19862	19914	19967	20019	20071	52
832	20123	20175	20228	20280	20332	20384	20436	20489	20541	20593	52
833	20645	20697	20749	20801	20853	20906	20958	21010	21062	21114	52
834	21166	21218	21270	21322	21374	21426	21478	21530	21582	21634	52
835	21686	21738	21790	21842	21894	21946	21998	22050	22102	22154	52
836	22206	22258	22310	22362	22414	22466	22518	22570	22622	22674	52
837	22725	22777	22829	22881	22933	22985	23037	23088	23140	23192	52
838	23244	23296	23348	23399	23451	23503	23555	23607	23658	23710	52
839	23762	23814	23865	23917	23969	24021	24072	24124	24176	24228	52
840	24279	24361	24383	24434	24486	24538	24589	24641	24693	24744	52
841	24796	24848	24899	24951	25002	25054	25106	25157	25209	25260	52
842	25312	25364	25415	25467	25518	25570	25621	25673	25724	25776	52
843	25828	25879	25931	25982	26034	26085	26137	26188	26239	26291	51
844	26342	26394	26445	26497	26548	26600	26651	26702	26754	26805	51
845	26857	26908	26959	27011	27062	27114	27165	27216	27268	27319	51
846	27370	27422	27473	27524	27576	27627	27678	27730	27781	27832	51
847	27883	27935	27986	28037	28088	28140	28191	28242	28293	28345	51
848	28396	28447	28498	28549	28601	28652	28703	28754	28805	28856	51
849	28908	28959	29010	29061	29112	29163	29214	29266	29317	29368	51
850	29419	29470	29521	29572	29623	29674	29725	29776	29827	29878	51
851	29930	29981	30032	30083	30134	30185	30236	30287	30338	30389	51
852	30440	30491	30541	30592	30643	30694	30745	30796	30847	30898	51
853	30949	31000	31051	31102	31153	31203	31254	31305	31356	31407	51
854	31458	31509	31560	31610	31661	31712	31763	31814	31864	31915	51
855	31966	32017	32068	32118	32169	32220	32271	32321	32372	32423	51
856	32474	32524	32575	32626	32677	32727	32778	32829	32879	32930	51
857	32981	33031	33082	33133	33183	33234	33285	33335	33386	33437	51
858	33487	33538	33588	33639	33690	33740	33791	33841	33892	33943	51
859	33993	34044	34094	34145	34195	34246	34296	34347	34397	34448	51
860	34498	34549	34599	34650	34700	34751	34801	34852	34902	34953	50
861	35003	35054	35104	35154	35205	35255	35306	35356	35406	35457	50
862	35507	35558	35608	35658	35709	35759	35809	35860	35910	35960	50
863	36011	36061	36111	36162	36212	36262	36313	36363	36413	36463	50
864	36514	36564	36614	36664	36715	36765	36815	36865	36916	36966	50
865	37016	37066	37116	37167	37217	37267	37317	37367	37418	37468	50
866	37518	37568	37618	37668	37718	37769	37819	37869	37919	37969	50
867	38019	38069	38119	38169	38219	38269	38319	38370	38420	38470	50
868	38520	38570	38620	38670	38720	38770	38820	38870	38920	38970	50
869	39020	39070	39120	39170	39220	39270	39319	39369	39419	39469	50
870	39519	39569	39619	39669	39719	39769	39819	39868	39918	39968	50
871	40018	40068	40118	40168	40218	40267	40317	40367	40417	40467	50
872	40516	40566	40616	40666	40716	40765	40815	40865	40915	40964	50
873	41014	41064	41114	41163	41213	41263	41313	41362	41412	41462	50
874	41511	41561	41611	41660	41710	41760	41809	41859	41909	41958	50
875	42008	42058	42107	42157	42206	42256	42306	42355	42405	42454	50
876	42504	42554	42603	42653	42702	42752	42801	42851	42900	42950	50
877	43000	43049	43099	43148	43198	43247	43297	43346	43396	43445	49
878	43494	43544	43593	43643	43692	43742	43791	43841	43890	43939	49
879	43989	44038	44088	44137	44186	44236	44285	44335	44384	44433	49
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No. 8800-9400. Log. 944483-973128.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
880	944483	944532	944581	944631	944680	944729	944779	944828	944877	944927	49
881	44976	45025	45074	45124	45173	45222	45272	45321	45370	45419	49
882	45469	45518	45567	45616	45665	45715	45764	45813	45862	45911	49
883	45961	46010	46059	46108	46157	46207	46256	46305	46354	46403	49
884	46452	46501	46550	46600	46649	46698	46747	46796	46845	46894	49
885	46943	46992	47041	47090	47139	47189	47238	47287	47336	47385	49
886	47434	47483	47532	47581	47630	47679	47728	47777	47826	47875	49
887	47924	47973	48021	48070	48119	48168	48217	48266	48315	48364	49
888	48413	48462	48511	48560	48608	48657	48706	48755	48804	48853	49
889	48902	48951	48999	49048	49097	49146	49195	49244	49292	49341	49
890	49390	49439	49488	49536	49585	49634	49683	49731	49780	49829	49
891	49878	49926	49975	50024	50073	50121	50170	50219	50267	50316	49
892	50365	50413	50462	50511	50560	50608	50657	50705	50754	50803	49
893	50851	50900	50949	50997	51046	51095	51143	51192	51240	51289	49
894	51337	51386	51435	51483	51532	51580	51629	51677	51726	51774	49
895	51823	51872	51920	51969	52017	52066	52114	52163	52211	52259	48
896	52308	52356	52405	52453	52502	52550	52599	52647	52696	52744	48
897	52792	52841	52889	52938	52986	53034	53083	53131	53180	53228	48
898	53276	53325	53373	53421	53470	53518	53566	53615	53663	53711	48
899	53760	53808	53856	53905	53953	54001	54049	54098	54146	54194	48
900	54242	54291	54339	54387	54435	54484	54532	54580	54628	54677	48
901	54725	54773	54821	54869	54918	54966	55014	55062	55110	55158	48
902	55206	55255	55303	55351	55399	55447	55495	55543	55592	55640	48
903	55688	55736	55784	55832	55880	55928	55976	56024	56072	56120	48
904	56168	56216	56264	56312	56361	56409	56457	56505	56553	56601	48
905	56649	56697	56745	56792	56840	56888	56936	56984	57032	57080	48
906	57128	57176	57224	57272	57320	57368	57416	57464	57511	57559	48
907	57607	57655	57703	57751	57799	57847	57894	57942	57990	58038	48
908	58086	58134	58181	58229	58277	58335	58373	58420	58468	58516	48
909	58564	58612	58659	58707	58755	58803	58850	58898	58946	58994	48
910	59041	59089	59137	59184	59232	59280	59328	59375	59423	59471	48
911	59518	59566	59614	59661	59709	59757	59804	59852	59900	59947	48
912	59995	60042	60090	60138	60185	60233	60281	60328	60376	60423	48
913	60471	60518	60566	60613	60661	60709	60756	60804	60851	60899	48
914	60946	60994	61041	61089	61136	61184	61231	61279	61326	61374	48
915	61421	61469	61516	61563	61611	61658	61706	61753	61801	61848	47
916	61895	61943	61990	62038	62085	62132	62180	62227	62275	62322	47
917	62369	62417	62464	62511	62559	62606	62653	62701	62748	62795	47
918	62843	62890	62937	62985	63032	63079	63126	63174	63221	63268	47
919	63315	63363	63410	63457	63504	63552	63599	63646	63693	63741	47
920	63788	63835	63882	63929	63977	64024	64071	64118	64165	64212	47
921	64260	64307	64354	64401	64448	64495	64542	64590	64637	64684	47
922	64731	64778	64825	64872	64919	64966	65013	65060	65108	65155	47
923	65202	65249	65296	65343	65390	65437	65484	65531	65578	65625	47
924	65672	65719	65766	65813	65860	65907	65954	66001	66048	66095	47
925	66142	66189	66236	66283	66329	66376	66423	66470	66517	66564	47
926	66611	66658	66705	66752	66798	66845	66892	66939	66986	67033	47
927	67080	67127	67173	67220	67267	67314	67361	67408	67454	67501	47
928	67548	67595	67642	67688	67735	67782	67829	67875	67922	67969	47
929	68016	68062	68109	68156	68203	68249	68296	68343	68389	68436	47
930	68483	68530	68576	68623	68670	68716	68763	68810	68856	68903	47
931	68950	68996	69043	69090	69136	69183	69229	69276	69323	69369	47
932	69416	69462	69509	69556	69602	69649	69695	69742	69788	69835	47
933	69882	69928	69975	70021	70068	70114	70161	70207	70254	70300	47
934	70347	70393	70440	70486	70533	70579	70626	70672	70719	70765	47
935	70812	70858	70904	70951	70997	71044	71090	71137	71183	71229	46
936	71276	71322	71369	71415	71461	71508	71554	71600	71647	71693	46
937	71740	71786	71832	71879	71925	71971	72018	72064	72110	72156	46
938	72203	72249	72295	72342	72388	72434	72480	72527	72573	72619	46
939	72666	72712	72758	72804	72851	72897	72943	72989	73035	73082	46
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No. 9400–10000. Log. 973128–000000.

No.	0	1	2	3	4	5	6	7	8	9	Diff.
940	973128	973174	973220	973266	973313	973359	973405	973451	973497	973543	46
941	73590	73636	73682	73728	73774	73820	73866	73913	73959	74005	46
942	74051	74097	74143	74189	74235	74281	74327	74373	74420	74466	46
943	74512	74558	74604	74650	74696	74742	74788	74834	74880	74926	46
944	74972	75018	75064	75110	75156	75202	75248	75294	75340	75386	46
945	75432	75478	75524	75570	75616	75661	75707	75753	75799	75845	46
946	75891	75937	75983	76029	76075	76121	76166	76212	76258	76304	46
947	76350	76396	76442	76487	76533	76579	76625	76671	76717	76762	46
948	76808	76854	76900	76946	76991	77037	77083	77129	77175	77220	46
949	77266	77312	77358	77403	77449	77495	77541	77586	77632	77678	46
950	77724	77769	77815	77861	77906	77952	77998	78043	78089	78135	46
951	78180	78226	78272	78317	78363	78409	78454	78500	78546	78591	46
952	78637	78683	78728	78774	78819	78865	78911	78956	79002	79047	46
953	79093	79138	79184	79230	79275	79321	79366	79412	79457	79503	46
954	79548	79594	79639	79685	79730	79776	79821	79867	79912	79958	46
955	80003	80049	80094	80140	80185	80231	80276	80322	80367	80412	45
956	80458	80503	80549	80594	80640	80685	80730	80776	80821	80867	45
957	80912	80957	81003	81048	81093	81139	81184	81229	81275	81320	45
958	81365	81411	81456	81501	81547	81592	81637	81683	81728	81773	45
959	81819	81864	81909	81954	82000	82045	82090	82135	82181	82226	45
960	82271	82316	82362	82407	82452	82497	82543	82588	82633	82678	45
961	82723	82769	82814	82859	82904	82949	82994	83040	83085	83130	45
962	83175	83220	83265	83310	83356	83401	83446	83491	83536	83581	45
963	83626	83671	83716	83762	83807	83852	83897	83942	83987	84032	45
964	84077	84122	84167	84212	84257	84302	84347	84392	84437	84482	45
965	84527	84572	84617	84662	84707	84752	84797	84842	84887	84932	45
966	84977	85022	85067	85112	85157	85202	85247	85292	85337	85382	45
967	85426	85471	85516	85561	85606	85651	85696	85741	85786	85830	45
968	85875	85920	85965	86010	86055	86100	86144	86189	86234	86279	45
969	86324	86369	86413	86458	86503	86548	86593	86637	86682	86727	45
970	86772	86816	86861	86906	86951	86995	87040	87085	87130	87174	45
971	87219	87264	87309	87353	87398	87443	87487	87532	87577	87622	45
972	87666	87711	87756	87800	87845	87890	87934	87979	88024	88068	45
973	88113	88157	88202	88247	88291	88336	88381	88425	88470	88514	45
974	88559	88603	88648	88693	88737	88782	88826	88871	88915	88960	45
975	89005	89049	89094	89138	89183	89227	89272	89316	89361	89405	45
976	89450	89494	89539	89583	89628	89672	89717	89761	89806	89850	44
977	89895	89939	89983	90028	90072	90117	90161	90206	90250	90294	44
978	90339	90383	90428	90472	90516	90561	90605	90650	90694	90738	44
979	90783	90827	90871	90916	90960	91004	91049	91093	91137	91182	44
980	91226	91270	91315	91359	91403	91448	91492	91536	91580	91625	44
981	91669	91713	91757	91802	91846	91890	91934	91979	92023	92067	44
982	92111	92156	92200	92244	92288	92333	92377	92421	92465	92509	44
983	92553	92598	92642	92686	92730	92774	92818	92863	92907	92951	44
984	92995	93039	93083	93127	93172	93216	93260	93304	93348	93392	44
985	93436	93480	93524	93568	93613	93657	93701	93745	93789	93833	44
986	93877	93921	93965	94009	94053	94097	94141	94185	94229	94273	44
987	94317	94361	94405	94449	94493	94537	94581	94625	94669	94713	44
988	94757	94801	94845	94889	94933	94977	95021	95064	95108	95152	44
989	95196	95240	95284	95328	95372	95416	95460	95504	95547	95591	44
990	95635	95679	95723	95767	95811	95854	95898	95942	95986	96030	44
991	96074	96117	96161	96205	96249	96293	96336	96380	96424	96468	44
992	96512	96555	96599	96643	96687	96730	96774	96818	96862	96905	44
993	96949	96993	97037	97080	97124	97168	97212	97255	97299	97343	44
994	97386	97430	97474	97517	97561	97605	97648	97692	97736	97779	44
995	97823	97867	97910	97954	97998	98041	98055	98128	98172	98216	44
996	98259	98303	98346	98390	98434	98477	98521	98564	98608	98652	44
997	98695	98739	98782	98826	98869	98913	98956	99000	99043	99087	44
998	99130	99174	99218	99261	99305	99348	99392	99435	99478	99522	44
999	99565	99609	99652	99696	99739	99783	99826	99870	99913	99957	43
	0	1	2	3	4	5	6	7	8	9	